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Minimal ℕ-System for Splitting Points of Stone-Čech Compactification

A Thesis

Submitted to the Council of the College of Science-University of Diyala in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

{يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ ۚ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ}

صدق اللَّهُ العلي العظيم
من سورة المجادلة ، آية (١١)

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I certify that this thesis entitled " **Minimal ℕ-System for Splitting points of Stone-Čech Compactification** " was prepared under my supervision at the Department of Mathematics Science\ College of Sciences\ University of Diyala by **Haider Mohammed Ridha**, as partial fulfillment of the requirements for the degree of Master of Science in Math Science

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Dedication

♦ To the teacher of humanity and the bearer of its message. . . . .

Our Prophet Mohammad (peace be upon him)

♦ To a soul that never left my soul. . . . . My beloved father (may God have mercy on you)

♦ To from heaven under her feet. . . . . My beloved mother (May God give you a long life)

♦ Who was standing with me after God, my dear wife

♦ To my liver. . . . . My children

♦ To the winds of my life. . . . . My sisters

To my colleagues, colleagues and everyone who helped. . . . .

Even if it was a prayer

I dedicate this humble effort

Haider Al-Qaisi

Aknolgment

Praise be to God, Lord of the Worlds, and prayers and peace be upon the master of the prophets and messengers, our Prophet Muhammad bin Abdullah, and upon the God of the pure and the pure.

Thanks, first to my Creator, Glory be to him, for his grace with which he enlightened us the path of knowledge and helped us to overcome difficulties and complete this research. To him be praise and thanks giving. I am pleased and honored to extend my sincere thanks and appreciation to Prof. Dr. Ali Hassan Nasser Al-Fayadh and Prof.Dr. Lieth Adul lateef Majed for them supervisions of this research and the great guidance they provided, which had the greatest impact in overcoming the difficulties and facilitating my task and for the sincere scientific sponsorship they showed throughout them supervision. They spared no effort in helping me and I sat with them for long hours. They does not find anything wrong with that, and they motivates me to research, desires me to do it, and strengthens my resolve to do so. Them experience was invaluable in formulating the most important points in my turn. I appreciate all of that which was mentioned, and God willing, I will be at my best, and that you will not feel afterwards them efforts have been wasted.

My thanks and appreciation go to my beloved mother, my wife and my brothers for their help, their standing by my side, and for bearing my burdens throughout the study period. May God preserve them for me and may I reward them well.

Haider 2022

Table of Contains

List of Abbreviations

Symbol	Definition
$\boldsymbol{K}$	Semi-group
$\boldsymbol{\mu}$	Filter
$\boldsymbol{D}$	Discrete set
$\boldsymbol{\lambda_x}$	Left translation
$\boldsymbol{\rho_x}$	Right translation
$\boldsymbol{\Lambda(K)}$	Topological center
$\boldsymbol{X^X}$	$f: X \to X$
$\boldsymbol{P(D)}$	Power set of nonempty set D
$\boldsymbol{\mathcal{L}_{(x)}}$	Orbit closure
$\mathbb{N}$	Set of natural numbers
$\boldsymbol{\beta}\mathbb{N}$	$\{ q : q$ is an ultra-filter on $\mathbb{N}\}$ Stone-Čech compactification of $\mathbb{N}$
$\widehat{\boldsymbol{\mathcal{M}}}$	$\{ q \in \beta\mathbb{N} : \mathcal{M} \in q\}$ base of topology on $\beta\mathbb{N}$
$\boldsymbol{\mathcal{P}_{f(\mathcal{B})}}$	$\{F : \emptyset \neq F \subseteq \mathcal{B}$, and F is finite$\}$
$\boldsymbol{M(\beta\mathbb{N})}$	Smallest ideal in $\beta\mathbb{N}$
$\mathcal{E}(\boldsymbol{K}, \boldsymbol{X})$	Enfolding semi-group
$\boldsymbol{\beta}\mathbb{N}^*$	$\beta\mathbb{N} \cup \{0\}$
$\boldsymbol{Y}$	$Y = \{0,1\}^{\mathbb{N}}$
$\boldsymbol{\omega}$	The set of Enfolding give $\mathcal{E}(\{T^n : n \in \mathbb{N}\}, Y)$
$\boldsymbol{\Gamma(B)}$	Is the phrase that χ_B the uniformly recurrent on $\{0.1\}^{\mathbb{N}}$

ABSTRACT

The major problem that we investigate in this work is the extent to which the full complexity of the Enfolding semi-group can be exhibited in a minimal system. We feel that the investigation of the Enfolding semi-group structure of a minimal system is a worthy one and interesting corresponding problems of a purely semi-group theoretic nature. We consider minimal left ideals $\mathcal{F}$ of the universal semi-group compactification βK of a topological semi-group K. We derive several conditions, some involving minimal system, which are equivalent to the ability to split q and r in this fashion, and then specialize to the case that $K = \mathbb{N}$, and the compactification is $\beta\mathbb{N}$. We give an application of a product discrete countable space of two-point with a specific system with several conditions, some involving minimal systems, which are equivalent to ability to splitting q and r in this way.

INTRODUCTION

In this thesis, we study properties in the theory of topological dynamical system. A **dynamical system** most likely originated at the end of the 19th century through the work of Henry Poincare in his study of celestial mechanics [3]. Mr. Joseph Auslander has conducted a number of studies on Minimal flows and their extensions see [1]. Many other study on automorphisms and equivalence relations in topological dynamics be done by David B. Ellis and Robert Ellis see [5]. However, one can say that dynamical systems draws its theory and techniques from many areas of mathematics, from analysis to geometry and topology, and into algebra [22]. We study the dynamical system with the **Stone-Čech** compactification of the set of natural number. This theory applied to get some results and give one application. For a purpose, we need to define the concepts of **Enfolding semi-group**. It proved to be a fundamental tool in the abstract theory of topological dynamical system. Consider a K-system with compact phase space X and K is a semi-group denoted by (K, X, α) where $\alpha: K \times X \to X$ is a continuous action of K on X, denoted by $\alpha(k,t) = k \cdot t = \alpha^k\,(t)$ [6] and [25]. The system (K, X) is called **minimal system** if the orbit $Kx = \{k \cdot x : k \in K\}$ is Dense in X for every $x \in X$ [17]. This equivalent to $\beta K \cdot x = X$ for every $x \in X$ for the extended action [5]. We look to K-system of closed left ideal $\mathcal{F}$ where $\mathcal{F}$ is a minimal left ideal of compact right topological semi-group H denoted by $(K, \mathcal{F})$ [22]. The major problem that we investigate is the extent of using the Enfolding semi-group can exhibited in a minimal system. In particular, we consider the question of whether $\beta\mathbb{N}$ can be the Enfolding semi-group of a minimal system. We show that the Enfolding semi-group of $\mathcal{F}$ is homeomorphically isomorphic to $\beta\mathbb{N}$ if and only if given $q \neq r$ in $\beta\mathbb{N}$, there is some p in the smallest ideal of $\beta\mathbb{N}$ with $q \cdot p \neq r \cdot p$.

In chapter one, we introduce the suitable notation, give an exposition of some of the elementary properties and the details of the construction. Plenty of these properties be applied directly, and serve in what follows in coming chapters and find some results.

Where we studied of the left (right) ideal, it is given as follows $\mathcal{F}$ (resp. $\mathcal{R}$) left and (right respectively) [12]. We look to the definition of the right topological semi-group denoted by $(K, ., \tau)$ where $(K, \cdot)$ is a semi-group, and (K, τ) is a topological space, and for all $x \in K$, $\rho_x : K \rightarrow K$ is continuous where as ρ_x is a right translation [2] and [11]. In addition, we study the concept of the filter and ultra-filter and various types of filter [21]. Many properties are given and then prove some of important properties. A filter μ on a set K is principal if there is a non-empty set $X \subseteq K$, such that $\mu = \{A \subseteq K : X \subseteq A\}$, otherwise μ is non-principal. We will define a topology on the set of all ultra-filters on a set $\mathbb{N}$, denoted by $\beta\mathbb{N} = \{ q : q \text{ is an ultra-filter on } \mathbb{N}\}$ is the set of all ultra-filter on a set $\mathbb{N}$ and establish some of the properties. We look to the algebraic structure on the set of ultra-filter by defined the operation + on it define as follow: for any $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we define $A - n = \{x \in \mathbb{N} : n + x \in A\}$ so for two ultra-filters $p, q \in \beta\mathbb{N}$ given then $p + q = \{A \subseteq \mathbb{N} | n \in \mathbb{N} | A - n \in q\} \in p\}$ see [8] and [13].

In chapter two, we are going to do an enlargement inside the space $\beta\mathbb{N}$ which is called stone-Čech compactification which is the technique for constructing a universal map from a topological space $\mathbb{N}$ [23] and [24], this is the largest compact space generated from the space. Where we have be proven the set $\mathcal{B}_q = \{ \overline{f[B]} : B \in q\}$ has finite intersection property where $f : \mathbb{N} \rightarrow W$ be continuous function where W be a compact space. While the finite intersection property is any sub collection family of sets we call this family satisfy finite intersection property if the intersection of any finite number of elements of this family is non-empty [8]. Moreover $(e, \beta N)$ is the stone-Čech compactification of $\mathbb{N}$ where $e : \mathbb{N} \rightarrow \beta\mathbb{N}$. We define the Enfolding semi-group denoted by $\mathcal{E}(T)$

where X be a compact Hausdorff topological space as follows: if $T = \{f : X \to X\}$ be a set of continuous function contained in X^X , then the closure of a set T is the Enfolding of T. In particular if (K, X, α) is a K-systems then the closure of the set $\{\alpha^k : k \in K\}$ in X^X denoted by $\mathcal{E}(K, X)$ will referred to the Enfolding semi-group of the K-systems. The Enfolding semi-group $\mathcal{E}(K, X)$ is a compact right-topological semi-group. In addition, algebraic properties was given for this concept [28].

In chapter three, we define the right M-stenography where M is a smallest ideal of the semi-group K given as: if $s \neq t \in K$ there is $q \in M$ such that $sq \neq tq$. In addition, the uuniformly recurrent and almost recurrent point was discussing in this chapter such that, let (K, X, α) be a system a point $x \in X$ is a uniformly recurrent point if given any neighbourhood V of x, there is a finite compact subset M of K such that given $k \in K$, there is $m \in M$ with $mkx \in V$ [7] and [9]. The uniformly recurrent points it will be exactly the points that are almost recurrent for the system $(\mathbb{N}, X)$ when $\mathbb{N}$ is given with the discrete topology. For our objective, we give an application by a particular example of a semi-group and its compactification with a particular system. Let $Y = \{0,1\}^{\mathbb{N}}$ consisting of two points discrete space and defined a shifting operator $T: Y \longrightarrow Y$ by $T(x)(n) = \chi(n+1)$where $x \in Y$.

The Publications

1- Utilization of Dynamical System on Enfolding Semi-group. Iraqi Journal of Science University of Baghdad – College of Science. Acceptance.

2- Isolation points of $\beta\mathbb{N}$ by using smallest ideal of $\beta\mathbb{N}$ with application. Neuro Quantology. June 2022. Volume 20. Issue 6. Page 1187-1191. doi:10.14704/nq.2022.6.20.NQ22114.

3- Space of Stone-Čech compactification $\beta\mathbb{N}$. University of Diyala, College of Science. Acceptance.

CHAPTER ONE

FUNDAMENTAL CONCEPTS WITH SOME RESULTS

1.1 Introduction

In this chapter, we have given a basic information, which are useful in our work. We defend the left (right) ideal on semi-group K this will lead us for the smallest ideal $M(K)$, which is the union of minimal left (or right) ideal. We explore the concepts of a filter and ultra-filters, which will be used to define the set $\beta\mathbb{N}$ for more information you can see [27]. In addition, several proofs.

1.2 Some Basic definitions and properties

Definition 1.2.1 [15]: A semi-group is a pair $(K,*)$ where K is non-empty set and $*$ is a associative binary operation on K.

Formally a binary operation on K is a function $*: K \times K \longrightarrow K$ such that the operation is associative iff $(p * q) * r = p * (q * r)$ for all p, q and r in K. Also K is closed under $*$ if $p * q \in K$ for any $p, q \in K$.

Example 1.2.2: Each of the following is a semi-group

1- The set of natural numbers $\mathbb{N}$ under multiplication or addition is a semi-group.

2- $(K,*)$ where K is a non-empty set where $x * y = x$ or y for all $x, y \in K$ is a semi-group.

3- $(\mathbb{N},\vee)$ such that $p \vee q = \max\{p, q\}$, where $p \in \mathbb{N}$ and $q \in \mathbb{N}$.

Definition 1.2.3 [14]: Let K be a semi-group, and let $\mathcal{F}, \mathcal{R}$ and I be a non-empty subset of K then:

1- $\mathcal{F}$ is a **left ideal** of K if and only if $\emptyset \neq \mathcal{F} \subseteq K$ and $K\mathcal{F} \subseteq \mathcal{F}$.

2- $\mathcal{R}$ is a **right ideal** of K if and only if $\emptyset \neq \mathcal{R} \subseteq K$ and $\mathcal{R}K \subseteq \mathcal{R}$.

3- I is an **ideal** of K if and only if I is a left ideal and right ideal of K.

Example 1.2.4 [20]:

1- Let K be a semi-group. If z is a zero in K then $\{z\}$ is an ideal in K.

2- In the commutative semi-group $(\mathbb{N}, +)$, the ideals are the sets $[a, \infty) = \{n \in \mathbb{N} : a \leq n\}$, where a is an arbitrary element of $\mathbb{N}$.

Definition 1.2.5 [14]: Let K be a semi-group, $\mathcal{R}$ is a right ideal of K, and $\mathcal{F}$ left ideal of K. Then

1- $\mathcal{F}$ is a **minimal left ideal** of K if and only if $\mathcal{F}$ is a left ideal of K and whenever J is a left ideal of K and $J \subseteq \mathcal{F}$ one has $J = \mathcal{F}$.

2- $\mathcal{R}$ is a **minimal right ideal** of K if and only if $\mathcal{R}$ is a right ideal of K and whenever J is a right ideal of K and $J \subseteq \mathcal{R}$ one has $J = \mathcal{R}$.

Example 1.2.6 [20]:

1- A Semi-groups with a zero has only one minimal left (right two-sided) ideal of K namely the trivial one $\{0\}$.

2- The integer numbers with addition $(Z,+)$ has no trivial minimal ideal.

The next Lemma can, be found in [11] as a problem, and will be given a proof for that.

Lemma 1.2.7 [11]: Let K be a semi-group.

1- Let $\mathcal{F}$ and L be left ideals of K. Then $\mathcal{F} \cap L$ is a left ideal of K if and only if $\mathcal{F} \cap L \neq \emptyset$.

2- Let $\mathcal{R}$ be a right ideal of K and let $\mathcal{F}$ be a left ideal of K. Then $\mathcal{R} \cap \mathcal{F} \neq \emptyset$.

Proof:

1- Suppose that $\mathcal{F} \cap L$ is a left ideal immediately by definition of left ideal we get $\mathcal{F} \cap L \neq \emptyset$. Conversely, suppose $\mathcal{F} \cap L \neq \emptyset$.

To show $\mathcal{F} \cap L$ left ideal we need to show $K(\mathcal{F} \cap L) \subseteq \mathcal{F} \cap L$. Let $x \in K(\mathcal{F} \cap L)$, so $x = ky$ where $k \in K$ and $y \in \mathcal{F} \cap L$. But $\mathcal{F}$ and L are left ideal then $ky \in \mathcal{F}$ and $ky \in L$, but $ky = x$ hance $x \in \mathcal{F} \cap L$.

2- Let $x \in \mathcal{R}$ and $y \in \mathcal{F}$ then $xy \in \mathcal{R}$ and $xy \in L$ by definition (1.2.3).

Lemma 1.2.8 [11]: Let K be a semi-group and let $t \in K$. Then Kt is a left ideal, tK is a right ideal and KtK is an ideal.

Proof: See the proof of Lemma (1.30 part a) in [11].

Theorem 1.2.9 [20]: Let K be a semi-group.

1- If $\mathcal{F}$ is a left ideal of K and $x \in \mathcal{F}$, then $Kx \subseteq \mathcal{F}$.

2- Let $\emptyset \neq \mathcal{F} \subseteq K$. Then $\mathcal{F}$ is a minimal left ideal of K if and only if for each $x \in \mathcal{F}$ implies $Kx = \mathcal{F}$.

Proof:

For part 1: This follows immediately from the definition of left ideal.

For part 2: Assume that $\mathcal{F}$ is a minimal left ideal of K and $x \in \mathcal{F}$. By Lemma (1.2.8) Kx is a left ideal and $Kx \subseteq \mathcal{F}$ by part (1) above. Since $\mathcal{F}$ is minimal left ideal, hence $Kx = \mathcal{F}$.

Conversely, Suppose $\mathcal{F}$ is a left ideal. Let L be a left ideal of K with $L \subseteq \mathcal{F}$. Pick $x \in L$. Then by part (1) above, $Kx \subseteq L$ and so $L \subseteq \mathcal{F} = Kx \subseteq L$. Hence $\mathcal{F}$ is a minimal.

Theorem 1.2.10 [11]: Let $\mathcal{F}$ be a minimal left ideal of the semi-group K, and let $J \subseteq K$. Then J is a minimal left ideal of K if and only if there is some $t \in K$ such that $J = \mathcal{F}t$.

Proof:

Assume J is a minimal left ideal of K and pick $t \in J$. Since $K\mathcal{F}t \subseteq \mathcal{F}t$ and $\mathcal{F}t \subseteq KJ \subseteq J$ then $\mathcal{F}t$ is left ideal of K in J. But, J is minimal and so $\mathcal{F}t = J$. Conversely, let $t \in K$ clearly $\mathcal{F}t \subseteq \mathcal{F}$ and a left ideal of K. But $\mathcal{F}$ is a minimal left ideal, so $\mathcal{F}t = \mathcal{F}$. Thus, $\mathcal{F}t$ is a minimal left ideal implies J is a minimal left ideal.

Corollary 1.2.11 [20]: Let K be a semi-group. If K has a minimal left ideal, then every left ideal of K contains a minimal left ideal.

Proof:

Let $\mathcal{F}$ be a minimal left ideal of K and let L be a left ideal of K. Pick $x \in L$. Then by Theorem (1.2.10), $\mathcal{F}x$ is a minimal left ideal which is contained in L.

Remark 1.2.12: We will denote of the **smallest ideal** set in the semi-group K by $M(K)$ which is the set contained in every ideal in K.

Theorem 1.2.13 [11]: Let K be a semi-group. If K has a minimal left ideal, then $M(K)$ exists and $M(K) = \cup \{\mathcal{F} : \mathcal{F}$ is a minimal left ideal of $K\}$.

Proof:

Let $\mathcal{H} = \cup \{\mathcal{F}: \mathcal{F}$ is a minimal left ideal of $K\}$. First we need to show that $\mathcal{H}$ is a minimal ideal. Let $\mathcal{F} \in \mathcal{H}$ be a minimal left ideal and let I be any ideal of K. By Lemma (1.2.7) part (2), $\mathcal{F} \cap I \neq \emptyset$. Let $s \in \mathcal{F} \cap I$ and $t \in K$, implies $ts \in \mathcal{F} \cap I$. So $\mathcal{F} \cap I$ is a left ideal and a subset of minimal left ideal $\mathcal{F}$. Therefore $\mathcal{F} \cap I = \mathcal{F}$. Since $\mathcal{F} \subseteq I$ hence $\mathcal{H} \subseteq I$, which implies that $\mathcal{H}$ is the smallest. It suffices to show that $\mathcal{H}$ is an ideal of K. Note that $\mathcal{H} \neq \emptyset$ by assumption. Let $s \in \mathcal{H}$ and pick a minimal left ideal $\mathcal{F}$ such that $s \in \mathcal{F}$. Then $ts \in \mathcal{F} \subseteq \mathcal{H}$, for all $t \in K$. Hence $\mathcal{H}$ is left ideal. By Theorem (1.2.10), $\mathcal{F}t$ is a minimal left ideal of K, so $\mathcal{F}t \subseteq \mathcal{H}$ while $st \in \mathcal{F}t$.

The next Lemma can, be found in [11] as a problem, and will be given a proof for that.

Lemma 1.2.14 [11]: Let K be a semi-group.

1- Let $\mathcal{F}$ be a left ideal of K. Then $\mathcal{F}$ is minimal if and only if $\mathcal{F}t = \mathcal{F}$ for every $t \in \mathcal{F}$.

2- Let J be an ideal of K. Then J is the smallest ideal if and only if $JtJ = J$ for each $t \in J$.

Proof:

1- If $\mathcal{F}$ is a minimal and $t \in \mathcal{F}$, then $\mathcal{F}t$ is a left ideal of K and $\mathcal{F}t \subseteq \mathcal{F}$, so $\mathcal{F}t = F$. Now assume $\mathcal{F}t = \mathcal{F}$ for every $t \in \mathcal{F}$ and let L be a left ideal of K with $L \subseteq \mathcal{F}$. Pick $t \in L$. Then $\mathcal{F} = \mathcal{F}t \subseteq \mathcal{F}L \subseteq L \subseteq \mathcal{F}$.

2- Suppose J is smallest ideal then $J = \cup$ {minimal left ideals}. Since J is an ideal, $t \in J$, then $JtJ \subseteq J$ by definition of ideal. Since J is smallest ideal then $J \subseteq JtJ$, hence $JtJ = J$. Conversely, suppose that $JtJ = J$, for each $t \in J$, to show J is the smallest ideal. Let I be an ideal of K such that $I \subseteq J$. Let $t \in I$, then is $t \in J$ implies $J = JtJ \subseteq JIJ \subseteq I \subseteq J$, and hence $J = JtJ$ is a minimal ideal. To show J is smallest ideal. Let I be an ideal of K to show $J \subseteq I$. Note that $I \cap J \neq \emptyset$, let $a \in I, b \in J$, this is $ab \in I$,and $ab \in J$. To show $J \cap I$ is an ideal, let $x \in J \cap I$,$t \in K$

$\Rightarrow tx \in J$ and $tx \in I$, hence $J \cap I$ is left ideal.

Similarly, $J \cap I$ is right ideal. This leads $J \cap I \subseteq J, I$

$$\Rightarrow J \cap I = J$$

$$\Rightarrow J \subseteq I$$

Hence J is the smallest ideal.

Theorem 1.2.15 [11]: Let K be a semigroup. If $\mathcal{F}$ is a minimal left ideal of K and $\mathcal{R}$ is a minimal right ideal of K, then $M(K) = \mathcal{FR}$.

Proof:

Clearly $\mathcal{FR}$ is an ideal of K. By Lemma (1.2.14 part 2), we need to show that $M(K) = \mathcal{FR}$. So, let $y \in \mathcal{FR}$. Then $\mathcal{FR}y\mathcal{F}$ is a left ideal of K which is contained in $\mathcal{F}$. So $\mathcal{FR}y\mathcal{F} = \mathcal{F}$ and hence $\mathcal{FR}y\mathcal{FR} = \mathcal{FR}$ since $\mathcal{F}$ is minimal left ideal.

Theorem 1.2.16 [11]: Let K be a semi-group and assume that there is a minimal left ideal of K which has an idempotent. Then every minimal left ideal has an idempotent.

Proof: See the proof of Theorem (1.56) in [11].

Definition 1.2.17 [18]: Let K be a semi-group and $x, y \in K$ we define the **left (resp. right) translations** on a function $\lambda_x: K \longrightarrow K$ (resp. $\rho_x: K \longrightarrow K$) as follows: $\lambda_x(y) = xy$ (resp. $\rho_x(y) = yx$).

Definitions 1.2.18 [11]:

1- The triple $(K, ., \tau)$ is called **right topological semi-group** where $(K, \cdot)$ is a semi-group, and (K, τ) is a topological space, if for all $x \in K$, $\rho_x : K \to K$ is continuous.

2- The triple $(K, ., \tau)$ is called **left topological Semi-group** where $(K, \cdot)$ is a semi-group, and (K, τ) is a topological space, if for all $x \in K$, $\lambda_x : K \to K$ is continuous

3- If the triple $(K, ., \tau)$ is a right topological semi-group and a left topological semi-group then $(K, ., \tau)$ is **a semi topological semi-group**.

4- The triple $(K,.,\tau)$ is **topological semi-group** where $(K,\cdot)$ is a semi-group, and (K,τ) is a topological space, if $\cdot : K \times K \to K$ is continuous.

Definition 1.2.19 [11]: We define the **topological center** of the semi-group K denoted by $\Lambda(K)$ which is define as follows: $\Lambda(K) = \{x \in K : \lambda_x$ is continuous$\}$, where K be a right topological semi-group.

Note 1.2.20: The center $\Lambda(K)$ is itself a semi-subgroup of K. Moreover $\Lambda(K) = K$ if and only if K is semi-topological semi-group.

Definition 1.2.21 [18]: A topological space $X$is Hausdorff (T_2-spaces) if for every $x, y \in X$ with $x \neq y$, there exist disjoint open subsets U, V of X such that $x \in U$ and $y \in V$.

Zorn's Lemma 1.2.22 [5]: If $(K, \leq)$ is a partially ordered set such that any increasing chain $k_1 \leq \cdots \leq k_i \leq \cdots$ has a supremum in K, then K itself has a maximal element.

The next theorem it is a fundamental important theorem that is related the compact right topological semi-group corresponding with the idempotent.

Theorem 1.2.23 [11]: Let K be a Hausdorff compact right topological semi-group. Then K contains at least one idempotent.

Proof:

Define the set $\mathcal{W} = \{Y \subseteq K : Y \neq \emptyset, Y$ is compact and $Y \cdot Y \subseteq Y\}$ which is the set of compact sub semi-groups of K. Note that $K \in \mathcal{W}$, So $\mathcal{W} \neq \emptyset$. Let $\mathcal{J}$ be a chain in $\mathcal{W}$. Since K is a Hausdorff consequently $\mathcal{J}$ is a collection of closed subsets from the compact space K. Hence, it has finite intersection property. So $\cap \mathcal{J} \neq \emptyset$ which is trivially compact and semi-group.

Implies $\cap \mathcal{J} \in \mathcal{W}$. So by Zorn's Lemma $\mathcal{W}$ has a minimal member say B. We need to show B is one member of $\mathcal{W}$.

Let $A = Bb$ where $b \in B$ then $A \neq \emptyset$. Since $A = \rho_b[B]$, then A is the continuous image of a compact space, hence it is compact.

Also $AA = BbBb \subseteq BBBb \subseteq Bb = A$, thus $A \in \mathcal{W}$. Since $A = Bb \subseteq BB \subseteq B$ and B is minimal of $\mathcal{W}$, so $A = B$. Let $C = \{x \in B: xb = b\}$. Note that $b \in B = Bb$, then $C \neq \emptyset$. Also, since $C = B \cap \rho_b^{-1}[\{b\}]$, so C is closed and implies its compact. Now given $x, c \in C$ one get $xc \in BB \subseteq B$ and $xcb = xb = b$ so $xc \in C$. Thus $C \in \mathcal{W}$. Since $C \subseteq B$ and B is minimal, Then $C = B$, so $b \in C$ and so $bb = b$.

Corollary 1.2.24 [11]: If K be a compact right topological semi-group. Then K has a minimal left ideal. More generally all-minimal left ideals in K will be closed and have an idempotent.

Proof:

Suppose $\mathcal{F}$ be a left ideal of K and let $x \in \mathcal{F}$. Since we have Hausdorff space, then $Kx = \rho_x(K)$ is a closed compact left ideal in $\mathcal{F}$. It follows any minimal left ideal is closed. By using the proof of Theorem (1.2.23), we have that any minimal left ideal has an idempotent. To complete the proof, we need to show that this satisfying for any left ideal of K contains a minimal left ideal. Let $\mathcal{F}$ be a left ideal of K and consider a set $\mathcal{H} = \{Y: Y \text{ is a closed left ideal of } K \text{ and } Y \subseteq \mathcal{F}\}$ which is partially ordered by inclusion. Note $\mathcal{H} \neq \emptyset$ since at least we have a left ideal Kx. Applying Zorn's Lemma, $\mathcal{H}$ has a minimal left ideals L. Since L is a minimal among these left closed ideals in $\mathcal{F}$, also since every left ideal contains a closed left ideal. Therefore L is a minimal left ideal.

1.3 A Filter and an Ultra-filter

Definition 1.3.1 [26]: Let K be any set, a **filter** on a set K is a non-empty set μ with the following properties:

1- $\emptyset \notin \mu$.

2- If $\mathcal{P}, q \in \mu$ then $\mathcal{P} \cap q \in \mu$.

3- If $\mathcal{P} \in \mu$ and $\mathcal{P} \subseteq q \subseteq K$ then $q \in \mu$.

Example 1.3.2: Consider the set μ to be a neighborhood of a point a in a topological space X. Then μ is a filter.

1- It's clearly that for any neighborhood of a point a say $q = (a-\in, a+\in)$ we have $\emptyset \notin \mu$.

2- Let $q = \left[a - \frac{\in}{2}, a + \frac{\in}{2}\right], \mathcal{P} = \left[a - \frac{\in}{4}, a + \frac{\in}{4}\right] \in \mu$ then

$$q \cap \mathcal{P} = \left[a - \frac{\in}{4}, a + \frac{\in}{4}\right] \in \mu.$$

3- Take $q = \left[a - \frac{\in}{4}, a + \frac{\in}{4}\right] \in \mu$, and $\mathcal{P} = \left[a - \frac{\in}{2}, a + \frac{\in}{2}\right]$ be a neighborhood for some point b, such that $\left[a - \frac{\in}{4}, a + \frac{\in}{4}\right] \subseteq \left[a - \frac{\in}{2}, a + \frac{\in}{2}\right] \subseteq K$ then $\left[a - \frac{\in}{2}, a + \frac{\in}{2}\right] \in \mu$.

Remarks 1.3.3 [4]:

1- The union of two filters on a set need not be a filter, for the counter example see example (2.1.4) (ii) in [4].

2- The intersection of all filters on K is the filter $\{K\}$ which is the weakest filter on K.

In the next following definition, we will introduce another important type of filter.

Definition 1.3.4 [26]: A filter μ on a set K is called an **ultra-filter** if it is not properly contained in any other filter on K.

Note 1.3.5 [8]: A filter μ on K is an **ultra-filter** if and only if for every $A \subseteq K$ either $A \in \mu$ or $A^c \in \mu$.

We record immediately the following very simple but also very useful fact about ultra-filters**.**

Example 1.3.6: 5: Let $K = \{a, b, c\}$

$\mu_1 = \{K\}$, $\mu_2 = \{\{a, b\}, K\}$, $\mu_3 = \{\{b, c\}, K\}$, $\mu_4 = \{\{c, a\}, K\}$,

$\mu_5 = \{\{a\}, \{a, b\}, \{a, c\}, K\}$, $\mu_6 = \{\{a, b\}, \{b, c\}, K\}$,

$\mu_7 = \{\{b\}, \{a, b\}, \{b, c\}, K\}$, $\mu_8 = \big\{\{c\}, \{c, a\}, \{b, c\}, K\big\}$

The filter μ_5, μ_7 and μ_8 are an ultrafilter on $K = \{a, b, c\}$ since there are no filter on K stecictly fine than μ_5, μ_7 and μ_8.

Remark 1.3.7 [11]: Let K be a set and let μ and υ be two ultra-filters on K. Then $\mu = \upsilon$ if and only if $\mu \subseteq \upsilon$.

Remark 1.3.8: Let K be a non-empty set and μ be a filter on K then by definition for every $q \subseteq K$ either $q \in \mu$ or $K \backslash q \in \mu$.

Definition 1.3.9 [26]: A filter μ on a set K is **principal** if there is a non-empty set $X \subseteq K$, such that $\mu = \{A \subseteq K : X \subseteq A\}$. Otherwise, μ is a non-principal.

Remark 1.3.10 [11]: Every ultra-filter on a finite set K is principal. Moreover, no principal ultra-filter is any ultra-filter on infinite set.

1.4 Topological Space on set $\beta\mathbb{N}$

In this section, we will define a topology on the set of ultra-filters on a special case for the set of natural number $\mathbb{N}$ and establish some of the properties. This will lead us to define the stone-Čech compactification space on $\mathbb{N}$.

Definition 1.4.1: Let $\mathbb{N}$ be a discrete topological space of natural number $\mathbb{N}$. We define the set of $\beta\mathbb{N} = \{ q : q$ is an ultra-filter on $\mathbb{N}\}$ that is the set of all ultra-filters on a set $\mathbb{N}$.

We will define a topology on the set of $\beta\mathbb{N}$ by describing a base explicitly and we shall be thinking of the ultra-filters as a point in this topology space $\beta\mathbb{N}$.

Definition 1.4.2: Let $\mathbb{N}$ be a discrete topological space we define the set.

1- For any $\mathcal{M}$ subset of $\mathbb{N}$, $\widehat{\mathcal{M}} = \{q \in \beta\mathbb{N} : \mathcal{M} \in q\}$, where q is an ultra-filter on $\mathbb{N}$.

2- Let $m \in \mathbb{N}$, then $e(m) = \{\mathcal{M} \subseteq \mathbb{N}, m \in \mathcal{M}\}$.

Lemma 1.4.3: For each $m \in \mathbb{N}$, $e(m)$ is the principal ultra-filter corresponding to m on the other word all ultra-filters generated from $\mathbb{N}$ are principal.

Proof:

First we need to show $e(m)$ is itself a filter.

1- Let $\mathcal{H}, \mathcal{B} \in e(m)$, then by definition of $e(m)$ above there is $m \in \mathcal{H}$ and $\mathcal{B} \in \mathcal{H}$ then $\mathcal{H} \cap \mathcal{B} \neq \emptyset$.

2- Let $\mathcal{H} \in e(m)$ and $\mathcal{B}$ be any set such that $\mathcal{H} \subseteq \mathcal{B} \subseteq \mathbb{N}$. Then $m \in \mathcal{H} \subseteq \mathcal{B}$. Immediately by the Definition (1.4.2) $\mathcal{B} \in e(m)$.

3- By definition of $e(m)$, we have $\emptyset \notin e(m)$.

Hence $e(m)$ is a filter, and by the Definition (1.4.2) $m \in \mathcal{H} \in e(m)$ then $m \notin \mathcal{H}^c$ implies $\mathcal{H}^c \notin e(m)$. Subsequently $e(m)$ is ultra-filter by Note (1.3.5), which is a principle.

In the next proposition, we illustrate some properties of the set we define above.

Proposition 1.4.4 [10]: For any two sets $\mathcal{M} \subseteq \mathbb{N}$ and $\mathcal{B} \subseteq \mathbb{N}$. $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{B}}$ have the following properties which are holds:

1- $\widehat{\mathcal{M}} = \emptyset$ if and only if $\mathcal{M} = \emptyset$.

2- $\widehat{\mathcal{M}} \subseteq \widehat{\mathcal{B}}$ if and only if $\mathcal{M} \subseteq \mathcal{B}$.

3- $\widehat{\mathcal{M} \cap \mathcal{B}} = \widehat{\mathcal{M}} \cap \widehat{\mathcal{B}}$.

4- $\widehat{\mathcal{M} \cup \mathcal{B}} = \widehat{\mathcal{M}} \cup \widehat{\mathcal{B}}$.

5- $\widehat{\mathcal{M}^c} = (\widehat{\mathcal{M}})^c$.

6- $(\widehat{\mathbb{N} / \mathcal{M}}) = \beta\mathbb{N} / \widehat{\mathcal{M}}$.

Proof:

1- Assume $\mathcal{M} \neq \emptyset$, there exists $m \in \mathcal{M}$. So $e(m)$ is a principle ultra-filter, leads to $\mathcal{M} \in e(m)$ this mean $\widehat{\mathcal{M}} \neq \emptyset$, a contradiction.

Conversely, suppose $\mathcal{M} = \emptyset$. Since $\emptyset \notin q$ for any ultra-filter, so $\widehat{\mathcal{M}} = \emptyset$.

2- If $\mathcal{M} \subseteq \mathcal{B}$, then for $q \in \widehat{\mathcal{M}}$, implies by definition $\mathcal{M} \in q$, so $\mathcal{B} \in q$. Hence $q \in \widehat{\mathcal{B}}$, so $\widehat{\mathcal{M}} \subseteq \widehat{\mathcal{B}}$.

Conversely, let $\widehat{\mathcal{M}} \subseteq \widehat{\mathcal{B}}$, and suppose $\mathcal{M} \nsubseteq \mathcal{B}$. Define $M^* = \mathcal{M} \backslash \mathcal{B} \neq \emptyset$. Choose some an ultra-filter q such that $M^* \in q$. Because $M^* \subseteq \mathcal{M}$, then $\mathcal{M} \in q$. Therefore $q \in \widehat{\mathcal{M}}$, which implies $q \in \widehat{\mathcal{B}}$, and hence $\mathcal{B} \in q$. But $\emptyset = M^* \cap \mathcal{B} \in q$, a contradiction.

3- Let $q \in \widehat{\mathcal{M} \cap \mathcal{B}}$, where q is an ultra-filter and therefor $\mathcal{M} \cap \mathcal{B} \in q$, which is equivalent $\mathcal{M} \in q$ and $\mathcal{B} \in q$. So by definition $q \in \widehat{\mathcal{M}}$ and $q \in \widehat{\mathcal{B}}$, this leads to be $q \in \widehat{\mathcal{M}} \cap \widehat{\mathcal{B}}$. Hence$\widehat{\mathcal{M} \cap \mathcal{B}} \subseteq \widehat{\mathcal{M}} \cap \widehat{\mathcal{B}}$. For the other direction, suppose $q \in \widehat{\mathcal{M}} \cap \widehat{\mathcal{B}}$, so $q \in \widehat{\mathcal{M}}$ and $q \in \widehat{\mathcal{B}}$, so by definition $\mathcal{M} \in q$ and $\mathcal{B} \in q$. Implies that $\mathcal{M} \cap \mathcal{B} \in q$ and so $q \in \widehat{\mathcal{M} \cap \mathcal{B}}$. Hence $\widehat{\mathcal{M}} \cap \widehat{\mathcal{B}} \subseteq \widehat{\mathcal{M} \cap \mathcal{B}}$. Therefore $\widehat{\mathcal{M} \cap \mathcal{B}} = \widehat{\mathcal{M}} \cap \widehat{\mathcal{B}}$.

4- Let $q \in \widehat{\mathcal{M} \cup \mathcal{B}}$, where q is an ultra-filter and so $\mathcal{M} \cup \mathcal{B} \in q$. Suppose $\mathcal{M}, \mathcal{B} \notin q$, so $\mathcal{M}^c \in q$ and $\mathcal{B}^c \in q$ this implies $\mathcal{M}^c \cap \mathcal{B}^c \in q$.

There fore $(\mathcal{M} \cup \mathcal{B}) \cap \mathcal{M}^c \cap \mathcal{B}^c = (\mathcal{M} \cup \mathcal{B}) \cap (\mathcal{M} \cup \mathcal{B})^c \in q$

$$= \emptyset \in q$$

That is a contradiction. Hence $\widehat{\mathcal{M} \cup \mathcal{B}} \subseteq \widehat{\mathcal{M}} \cup \widehat{\mathcal{B}}$. Other direction, let $q \in \widehat{\mathcal{M}} \cup \widehat{\mathcal{B}}$, then $q \in \widehat{\mathcal{M}}$ or $q \in \widehat{\mathcal{B}}$. Suppose $q \in \widehat{\mathcal{M}}$, $\mathcal{M} \in q$.

Since $\mathcal{M} \subseteq \mathcal{M} \cup \mathcal{B} \subseteq \mathbb{N}$, by definition of filter $\mathcal{M} \cup \mathcal{B} \in q$, so $q \in \widehat{\mathcal{M} \cup \mathcal{B}}$. Similarly, if $q \in \widehat{\mathcal{B}}$. Hence $\widehat{\mathcal{M} \cup \mathcal{B}} = \widehat{\mathcal{M}} \cup \widehat{\mathcal{B}}$.

5- For $q \in \widehat{\mathcal{M}^c}$, so $\mathcal{M}^c \in q$ and hence $\mathcal{M} \notin q$. This mean $q \in (\widehat{\mathcal{M}})^c$ and so $\widehat{\mathcal{M}^c} \subseteq \left(\widehat{\mathcal{M}}\right)^c$. Now for $q \in (\widehat{\mathcal{M}})^c$, $q \notin \widehat{\mathcal{M}}$. This mean $\mathcal{M} \notin q$ so $\mathcal{M}^c \in q$. So $q \in \widehat{\mathcal{M}^c}$ and $\left(\widehat{\mathcal{M}}\right)^c \subseteq \widehat{\mathcal{M}^c}$. Therefore $\widehat{\mathcal{M}^c} = (\widehat{\mathcal{M}})^c$.

6- Note $(\widehat{\mathbb{N} / \mathcal{M}}) = \{q \in \beta\mathbb{N} : \mathcal{M}^c \in q\}$, this leads to $\mathcal{M} \notin q$ and so $q \notin \widehat{\mathcal{M}}$ implies $q \in \beta\mathbb{N}/\widehat{\mathcal{M}}$. Conversely, let $q \in \beta\mathbb{N}/\widehat{\mathcal{M}} = \{q \in \beta\mathbb{N} : \mathcal{M} \notin q\}$. So $\mathcal{M} \notin q$ implies that $\mathcal{M}^c \in q$, then $q \in (\widehat{\mathbb{N} / \mathcal{M}})$.

Theorem 1.4.5 [11]: Let D be a discrete set and let A be a subset of $P(D)$ which has the finite intersection property. Then there is an ultra-filter μ on D such that $A \subseteq \mu$.

The next theorem it's in [11] but we will reprove it for our purpose in special case with the discrete set of natural numbers.

Theorem 1.4.6: Let $\mathbb{N}$ a set of natural numbers, then $\beta\mathbb{N}$ is a compact Hausdorff space.

Proof:

Suppose that $\mathcal{P}$ and q be two distinct ultra-filter elements of $\beta\mathbb{N}$. If $\mathcal{M} \in \mathcal{P} \backslash q$ then $\mathcal{M}^c = \mathbb{N} \backslash \mathcal{M} \in q$. Then $\widehat{\mathcal{M}}$ and $\widehat{\mathbb{N} / \mathcal{M}}$ are disjoint open sets subsets of $\beta\mathbb{N}$ containing $\mathcal{P}$ and q respectively. Hence, $\beta\mathbb{N}$ is Hausdorff space. Now, for compactness of $\beta\mathbb{N}$ we need to show that every collection of closed sets of $\mathbb{N}$ satisfies the finite intersection property has non- empty intersection.

Note that the set of the form $\widehat{\mathcal{M}}$ is the stone set $\mathcal{M}$ which acts as both an open and closed sets bases because $(\widehat{\mathbb{N} / \mathcal{M}}) = \beta\mathbb{N} / \widehat{\mathcal{M}}$.

To prove $\beta\mathbb{N}$ is a compact we will show that the family $\mathcal{H} = \{$the stone set $\widehat{\mathcal{M}}$ with finite intersection property$\}$ has non-empty intersection. Let $\mathcal{B} = \{\mathcal{M} \subseteq \mathbb{N}: \widehat{\mathcal{M}} \in \mathcal{H}\}$. If $F \in \wp_{f(\mathcal{B})} = \{F: \emptyset \neq F_i \subseteq \mathcal{B}, i = 1,2, \ldots, n$, and F is finite$\}$, this mean for each $\mathcal{M}_i \in F_i$, $\widehat{\mathcal{M}}_i \in \mathcal{H}$.

From definition of $\mathcal{H}$ there is some $\mathcal{P} \in \cap_{\mathcal{M} \in F} \widehat{\mathcal{M}}$, and by definition of $\widehat{\mathcal{M}}$ we get $\cap F_i \in \mathcal{P}$. Thus $\cap F_i \neq \emptyset$ and hence $\mathcal{B}$ has finite intersection property. So by Theorem (1.4.5) there is an ultra-filter $q \in \beta\mathbb{N}$ such that $\mathcal{B} \subseteq q$, and so $q \in \cap \mathcal{H}$. Therefore $\beta\mathbb{N}$ is compact.

Lemma 1.4.7 [11]: The set of the form $\widehat{\mathcal{M}}$ are the clopen subset of $\beta\mathbb{N}$.

Proof:

Based on the previous theorem, we say $\widehat{\mathcal{M}}$ is a base of open and closed set in $\beta\mathbb{N}$, so $\widehat{\mathcal{M}}$ is clopen. We will try to show that any clopen subset of $\beta\mathbb{N}$ belongs to this kind of family. Let B be any clopen subset of $\beta\mathbb{N}$. Let $\mathcal{H} = \{\widehat{\mathcal{M}}: \mathcal{M} \subseteq \mathbb{N}$ and $\widehat{\mathcal{M}} \subseteq B\}$. Since $\mathcal{H}$ is a collection of a basis of an open set $\widehat{\mathcal{M}}$, so $\mathcal{H}$ is open cover of B, but B is closed subset of a compact space $\beta\mathbb{N}$. Therefore, B is a compact. Pick a finite subfamily $\mathcal{J}$ of $\mathbb{N}$ such that $B = \cup_{\mathcal{M} \in \mathcal{J}} \widehat{\mathcal{M}}$, so by using Proposition (1.4.4 part 4) we have $B = \widehat{\cup \mathcal{M}}$.

Theorem 1.4.8: A map $e: \mathbb{N} \longrightarrow \beta\mathbb{N}$ where e is a one to one and $e[\mathbb{N}]$ is a dense subset of $\beta\mathbb{N}$ which its points are precisely the isolated points of $\beta\mathbb{N}$.

Proof:

Let $a \neq b \in \mathbb{N}$, so $b \notin \{a\}$ implies $\{a\} \notin e(b)$, but, $e(b)$ is an ultra-filter then $\{a\}^c = \mathbb{N}\setminus \{a\} \in e(b)\setminus e(a)$. Hence $e(a) \neq e(b)$. Therefore, e is one to one.

To show $e[\mathbb{N}]$ is a dense subset on $\beta\mathbb{N}$. We need to show $e[\mathbb{N}]$ has its point and its limit point of $e[\mathbb{N}]$. So we will try to show if $\mathcal{P}$ is a point in $\beta\mathbb{N}$ is a limit point of $e[\mathbb{N}]$ if every neighborhood of $\mathcal{P}$ contains at last one point of $e[\mathbb{N}]$ different from $\mathcal{P}$ itself. Let $\widehat{\mathcal{M}}$ be a basic open subset of $\beta\mathbb{N}$, by definition of $\widehat{\mathcal{M}}$ then $\mathcal{M} \neq \emptyset$. Note that any $a \in \mathcal{M}$ satisfy $e(a) \in e[\mathbb{N}] \cap \widehat{\mathcal{M}}$ and so $e[\mathbb{N}] \cap \widehat{\mathcal{M}} \neq \emptyset$ which mean $e[\mathbb{N}]$ has its limit point. Hence $e[\mathbb{N}]$ is a dense in $\beta\mathbb{N}$.

Finally, to show the points of $\beta\mathbb{N}$ are isolated. Note for any $a \in \mathbb{N}$, $e(a)$ is isolated in $\beta\mathbb{N}$ because $\{e(a)\} = \{\hat{a}\}$ is an open subset of $\beta\mathbb{N}$ whose only member is $e(a)$. Then by definition of isolated then $e(a)$ is an isolated point.

The following definition is defined an operation on the set $\beta\mathbb{N}$ that give the algebraic structure for our study.

Definition 1.4.9 [20]: Let $(\mathbb{N}, +)$ is a semi-group, for any $B \subseteq \mathbb{N}$ and $m \in \mathbb{N}$ we define $B - m = \{x \in \mathbb{N}: m + x \in B\}$. For any two ultra-filters $\mathcal{P}$ and q we define their sum by:

$$\mathcal{P} + q = \{B \subseteq \mathbb{N} | \{m \in \mathbb{N}: B - m \in q\} \in \mathcal{P}\}.$$

Example 1.4.10: Consider the two principle ultra-filters $\mathcal{P}(m_1)$ and $\mathcal{P}(m_2)$, we will show that $\mathcal{P}(m_1)+\mathcal{P}(m_2)=\mathcal{P}(m_1+m_2)$.

Solution:

For any $B\in\mathcal{P}(m_1)+\mathcal{P}(m_2)$, since$\{m\in\mathbb{N}: B-m\in\mathcal{P}(m_2)\}\in\mathcal{P}(m_1)$ by definition $m_1\in\{m\in\mathbb{N}: B-m\in\mathcal{P}(m_2)\}$, since $B-m_1\in\mathcal{P}(m_2)$ leads to $m_2\in B-m_1$. Hence, $m_1+m_2\in B$. So, $B\in\mathcal{P}(m_1+m_2)$.

The other direction $B\in\mathcal{P}(m_1+m_2)$ by definition $m_1+m_2\in B$. We get $m_2\in B-m_1$.So, $B-m_1\in\mathcal{P}(m_2)$. This leads to have $m_1\in\{m\in\mathbb{N}: B-m\in\mathcal{P}(m_2)\}$, then $\{m\in\mathbb{N}: B-m\in\mathcal{P}(m_2)\}\in\mathcal{P}(m_1)$. Hence, $B\in\mathcal{P}(m_1)+\mathcal{P}(m_2)$.

Lemma 1.4.11: The operation + that is defined on $\beta\mathbb{N}$ is a binary operation as well as associative.

Proof:

Let $\mathcal{P}$ and $q\in\beta\mathbb{N}$. First, we need to show $\mathcal{P}+q\in\beta\mathbb{N}$. By showing $\mathcal{P}+q$ is an ultra-filter. It is clear that by definition $\emptyset\notin(\mathcal{P}+q)$ and $\mathbb{N}\in(\mathcal{P}+q)$. Suppose $A,B\in(\mathcal{P}+q)$. Then $A\cap B-m\in\mathcal{P}$ iff $A-m\in\mathcal{P}$ and $B-m\in\mathcal{P}$. Therefore, $\{m\in\mathbb{N}:(A\cap B-m)\in\mathcal{P}\}=\{m\in\mathbb{N}: A-m\in\mathcal{P}\}\cap\{m\in\mathbb{N}: B-m\in\mathcal{P}\}$. But $A,B\in\mathcal{P}+q$, so $\{m\in\mathbb{N}: A-m\in\mathcal{P}\}\in q$ and $\{m\in\mathbb{N}: B-m\in\mathcal{P}\}\in q$, we get $\{m\in\mathbb{N}:(A\cap B-m)\in\mathcal{P}\}\in q$ and $A\cap B\in\mathcal{P}+q$. Suppose $A\in\mathcal{P}+q$ and $A\subseteq\mathcal{C}\subseteq\mathbb{N}$. Then $A-m\subseteq\mathcal{C}-m$ for all $m\in\mathbb{N}$. So $A-m\in\mathcal{P}$ implies $\mathcal{C}-m\in\mathcal{P}$. Thus $\{m\in\mathbb{N}: A-m\in\mathcal{P}\}\subseteq\{m\in\mathbb{N}:\mathcal{C}-m\in\mathcal{P}\}$ and $\{m\in\mathbb{N}:\mathcal{C}-m\in\mathcal{P}\}\in q$. Therefore $\mathcal{C}\in\mathcal{P}+q$. We want to show that $\mathcal{P}+q$ is an ultra-filter, let $A\subseteq\mathbb{N}$ with $A\notin\mathcal{P}+q$.

Thus, $\{ m \in \mathbb{N}: A - m \in \mathcal{P}\} \notin q$ and so $\{ m \in \mathbb{N}: A - m \notin \mathcal{P}\} \in q$. So we will get $\{ m \in \mathbb{N}: A^c - m \in \mathcal{P}\} = \{ m \in \mathbb{N}: (A - m)^c \in \mathcal{P}\} = \{ m \in \mathbb{N}: A - m \in \mathcal{P}\}^c \in q$. Therefor, $A^c \in \mathcal{P} + q$. This shows that $\mathcal{P} + q$ is an ultra-filter.

Now, for associative part for our operation.

Let $A \subseteq \mathbb{N}$. Then $A \in \mathcal{P} + (q + r)$ iff $\{ m \in \mathbb{N}: (A - m) \in \mathcal{P}\} \in q + r$ and for more clarity iff $\{n \in \mathbb{N}: (\{m \in \mathbb{N}: (A - m) \in \mathcal{P} \} - n\} \in q \} \in r$. Note $\{m \in \mathbb{N}: (A - m) \in \mathcal{P} \} - n = \{m - n: (A - m) \in \mathcal{P} \}$

$$= \{m \in \mathbb{N}: ((A - n) - m) \in \mathcal{P} \}.$$

So, $A \in \mathcal{P} + (q + r) \Leftrightarrow \{n \in \mathbb{N}: ((A - n) - m) \in \mathcal{P} \} \in q \} \in r$ iff $\{n \in \mathbb{N}: (A - n) \in \mathcal{P} + q \} \in r$ iff $A \in (\mathcal{P} + q) + r$.

Theorem 1.4.12: For any $\mathcal{P} \in \beta\mathbb{N}$, the map $\lambda_{\mathcal{P}}: \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$ that is given by $\lambda_{\mathcal{P}}(q) = \mathcal{P} + q$ is continuous.

Proof:

We show that the inverse image of a basic open set is open. Indeed,

$$\begin{aligned} {\lambda_{\mathcal{P}}}^{-1}\left(\widehat{\mathcal{M}}\right) &= \{q \in \beta\mathbb{N}: \lambda_{\mathcal{P}}(q) \in \widehat{\mathcal{M}}\} \\ &= \{q : \mathcal{P} + q \in \widehat{\mathcal{M}} \} \\ &= \{q : \mathcal{M} \in \mathcal{P} + q\} \\ &= \{q : \{n : \mathcal{M} - n \in \mathcal{P}\} \in q\} \\ &= \{n : \widehat{\mathcal{M} - n} \in \mathcal{P}\} . \end{aligned}$$

Therefore, that $\lambda_{\mathcal{P}}$ is continuous.

Lemma 1.4.13 [8]: (Idempotent lemma) There is element $\mathcal{P} \in \beta\mathbb{N}$ with $\mathcal{P} + \mathcal{P} = \mathcal{P}$.

Proof:

Suppose that $\mathcal{B}$ be the set of compact semi-groups which are contained in $\beta\mathbb{N}$ .Because $\beta\mathbb{N} \in \mathcal{B}$ is non-empty. By inclusion, it is partially ordered. Every chain A has $\bigcap_{A\in A} A$ as a non-empty lower bound (it is compact and non-empty since all the A 's are compact and it is easy to be a semi-group). By Zorn's lemma we have a minimal compact semi-group L. We claim that any $\mathcal{P} \in L$ is idempotent.

We first observe that $L + \mathcal{P}$ is a compact (by left continuity of addition semi-group). If $\mathcal{P}_1 + \mathcal{P}$ and $\mathcal{P}_2 + \mathcal{P}$ are elements of $L + \mathcal{P}$ then so is $(\mathcal{P}_1 + \mathcal{P}) + (\mathcal{P}_2 + \mathcal{P}) = (\mathcal{P}_1 + \mathcal{P} + \mathcal{P}_2) + q$. Since $L + \mathcal{P} \subseteq$ L we have $L + \mathcal{P} = L$ by minimality. Now set C={ $q \in L$: $q + \mathcal{P} = \mathcal{P}$ } because $L = L + \mathcal{P}, C$ is non-empty. It is compact by continuity also it is a semi-group: $q_1 + \mathcal{P} = \mathcal{P}$ and $q_2 + \mathcal{P} = \mathcal{P}$ imply $(q_1 + q_2) + \mathcal{P} = \mathcal{P}$, since C $\subseteq L$ by minimality of L, in fact $C = L$ so $\mathcal{P} \in C$ and $\mathcal{P} + \mathcal{P} = \mathcal{P}$.

CHAPTER TWO

STONE-ČECH COMPACTIFICATION OF βN AND K-SYSTEM WITH ENFOLDING SEMI-GROUP

2.1 Introduction

In this chapter we briefly study the **stone-Čech compactification**, which is the largest compact space generated from the space. We will define it in the case of the set of natural numbers $\mathbb{N}$, and which is denoted by $\beta\mathbb{N}$. That is, $\beta\mathbb{N}$ is the set of all ultra-filters on $\mathbb{N}$. The algebraic and topological properties of $\beta\mathbb{N}$ will be used to understand and study the dynamical behavior of the system and present several of its applications. In addition, we introduce the concept of **Enfolding semi-group** and its theory, which we use it in the theory of topological dynamics. It reflects various properties of a dynamical system. We give some properties concerning its structure. We describe the connections between the algebraic and topological properties of the Enfolding semi-group and various properties. In addition, we introduce the concepts of ***K*-system** and theoretical properties related to them.

2.2 Stone-Čech compactification $\boldsymbol{\beta}\mathbb{N}$

Definition 2.2.1[19]: The stone-Čech Compactification of Discrete topological space $\mathbb{D}$ is a pair $(\varphi, \mathbb{Y})$ such that:

1) $\mathbb{Y}$ is a compact space.

2) $\mathbb{D}$ Embedding into $\mathbb{Y}$ by φ.

3) $\varphi[\mathbb{D}]$ is dense in $\mathbb{Y}$, and

4) Given any compact space W and any continuous function $f: \mathbb{D} \longrightarrow W$ there is a continuous function $g: \mathbb{Y} \longrightarrow W$ such that $g \, o \, \varphi = f$.

$$\begin{tikzcd} & \mathbb{Y} \arrow[dr, "g"] & \\ \mathbb{D} \arrow[ur, "\varphi"] \arrow[rr, "f"'] & & W \end{tikzcd}$$

Remark 2.2.2[19]:

1) Any two Stone-Čech Compactifications of the same topological space $\mathbb{D}$ are homeomorphism.

2) The topology induced on $\mathbb{D}$ as a subset of $\mathbb{Y}$ is the original topology of $\mathbb{D}$.

Note 2.2.3: We will concentrate our work on a special case when $\mathbb{D} = \mathbb{N}$, and next theorem shows that $\beta\mathbb{N}$ is the stone-Čech compactification corresponding to $\mathbb{N}$.

The next proposition can, be found in [11] as a problem, and will be given a proof for that. The importance of this proposition is that is one of the properties for $\beta\mathbb{N}$ related to the finite intersection property.

Proposition 2.2.4: Let $f: \mathbb{N} \to W$ be a continuous function where W be a compact space. Then $\mathcal{B}_{q} = \{ \overline{f[B]}: B \in q\}$ has finite intersection property for any $q \in \beta\ \mathbb{N}$.

Proof:

Pick B_1,B_2, …,$B_n \in q$. If we can show that $\cap_{j=1}^{n} \overline{f(B_j)} \neq \emptyset$ then we are finished.

Because B_1,B_2,B_3,…,$B_n \in q$ hence $\emptyset \neq f(\cap_{j=1}^{n} B_j)$

$$\subseteq \cap_{j=1}^{n} f\left(B_j\right)$$

$$\subseteq \cap_{j=1}^{n} f(\overline{B_j}) \quad \{ f \text{ is continuous}\}$$

$$\subseteq \overline{\cap_{j=1}^{n} f(B_j)} \quad \text{since } f(\overline{B_j}) \subseteq \overline{f(B_j)}.$$

Theorem 2.2.5: If $\mathbb{N}$ is a discrete space, then the pair $(e, \beta N)$ is the stone-Čech compactification of $\mathbb{N}$.

Proof:

We need to achieve the stone-Čech compactification conditions, which are:

1) $\beta\mathbb{N}$ is compact, this was proved in the Theorem (1.4.6).

2) To show e is an embedding:

i) We claim that $e: \mathbb{N} \to \beta\mathbb{N}$ is injective. Let $t \neq d \in \mathbb{N}$. By definition $e(t)$ and $e(d)$ are two ultra-filters generated by t and d respectively. So $\{t\}^c = \mathbb{N} \backslash \{t\} \in e(d) \backslash e(t)$. Hence, $e(t) \neq e(d)$ i.e we have a one to one condition.

ii) Obviously, e is continuous, because $\mathbb{N}$ is discrete space.

iii) If we can show that e is a closed map we are done. Suppose $B \subseteq \mathbb{N}$ be a closed subset and since $\hat{B} \subseteq e[B]$ then $e[B] \cap \hat{B} \subseteq e[B]$. Also, $e[B] \subseteq e[B] \cap \hat{B}$, since if $\mathcal{t} \in e[B] \Rightarrow \mathcal{t} = (\ b') = e(\ b')$ where $b' \in B \Rightarrow B \in \mathcal{t} \Rightarrow \mathcal{t} \in \hat{B}$.

Implies $e[B] = e[B] \cap \hat{B}$.

3) To show $e[\mathbb{N}]$ is a dense. We need to show $e[\mathbb{N}]$ has its points and a limit point of $e[\mathbb{N}]$. So, we will try to show that if $\mathcal{p}$ is a point in $\beta\mathbb{N}$ which is a limit point of $e[\mathbb{N}]$, then every neighborhood of $\mathcal{p}$ contains at least one point of $e[\mathbb{N}]$ deferent from $\mathcal{p}$ itself. Let $\hat{A}$ be a basic open subset of $\beta\mathbb{N}$, then $A \neq \emptyset$, any $a \in A$ satisfy $e(a) \in e[\mathbb{N}] \cap \hat{A}$ and so $e[\mathbb{N}] \cap \hat{A} \neq \emptyset$ i.e. $e[\mathbb{N}]$ has its limit point. Hence, $e[\mathbb{N}]$ is a dance in $\beta\mathbb{N}$.

4) Given a compact space W and let $f: \mathbb{N} \longrightarrow W$ be continuous, to show there is a continuous function $g: \beta\mathbb{N} \longrightarrow W$ such that it has a commutative diagram.

$$\begin{tikzcd} & \beta\mathbb{N} \arrow[dr, "g"] & \\ \mathbb{N} \arrow[ur, "e"] \arrow[rr, "f"'] & & W \end{tikzcd}$$

First, we need to define the function g. For each $\mathcal{q} \in \beta\mathbb{N}$, let $\mathcal{A}_{\mathcal{q}} = \{\ \overline{f[B]}: B \in \mathcal{q}\}$. So for each $\mathcal{q} \in \beta\mathbb{N}$, by Proposition (2.2.4) $\mathcal{A}_{\mathcal{q}}$has the finite intersection property, and because W is compact, so $\mathcal{A}_{\mathcal{q}}$ has non-empty intersection. Choose $g(\mathcal{q}) \in \cap\, \mathcal{A}_{\mathcal{q}}$.

Secondly, to show the diagram is commutative. Let $n \in \mathbb{N}$ then $\{n\} \in e(n) = \{B \subseteq \mathbb{N}: n \in B\}$.

So, $g\big(e(n)\big) \in \overline{f[\{n\}]} = \overline{[\{f(n)\}]}$

$$= \{f(n)\} \text{ since singleton is closed.}$$

Immediately by definition $g\ o\ e = f$.

Finally, to show g is continuous. Let $q \in \beta\mathbb{N}$ and let v be a neighborhood of $g(q)$ in W, and since W is compact Hausdorff space then, W is regular space. So pick a neighborhood u of $g(q)$ with $\bar{u} \subseteq v$ by definition of regular we get a closed set.

Let $B = f^{-1}[u] \in \mathbb{N}$, we claim $B \in q$, suppose $\mathbb{N}\backslash B \in q$ then $g(q) \in \overline{f[\mathbb{N}\backslash B]}$, and since u is a neighborhood of $g(q)$. So $u \cap f[\mathbb{N}\backslash B] \neq \emptyset$ that is a contradiction because $B = f^{-1}[u]$. Hence $B \in q$ then $q \in \hat{B}$ is a neighborhood of q. Claim $g[\hat{B}] \subseteq v$. Let $\mathcal{P} \in \hat{B} = \bar{B}$ and suppose $g(\mathcal{P}) \notin v$, then $W\backslash\,\bar{u}$ is a neighborhood of $g(\mathcal{P})$ and $g(\mathcal{P}) \in \overline{f[B]}$, since $\mathcal{P} \in \hat{B} = \bar{B}$ then $f(\mathcal{P}) \in f(\bar{B}) \subseteq \overline{f(B)}$ so $(W\backslash\bar{u}) \cap f[B] \neq \emptyset$ that is a contradiction since $B = f^{-1}[u]$.

The next proposition, which is founded as an open problem, we found in [11]. The proof we consider is that if the two continuous maps identify on $e[\mathbb{N}]$ then they will be equal.

Proposition 2.2.6: Let $\mathcal{B}_{q} = \{\overline{f[A]}: A \in q\}$ be a set belong to W. For each $q \in \beta\mathbb{N}$, we have $\bigcap \mathcal{B}_{q}$ is a singleton.

Proof:

By proposition (2.2.4) we show that $\mathcal{B}_{q}$ has a finite intersection property and since W is compact, then every family of closed subsets having the finite intersection property has non-empty intersection. So, $\bigcap\mathcal{B}_{q} \neq \emptyset$, hence, there exists $w \in \bigcap\mathcal{B}_{q}$ such that w is an element of all $\overline{f(A)}$ for all $A \in q$. Now to show $\bigcap\mathcal{B}_{q}$ is singleton.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "f"] \arrow[d] & W \\ \beta\mathbb{N} \arrow[ur, "g"'] & \end{tikzcd}$$

Choose $w = g(q) \in \bigcap\mathcal{B}_{q} = \bigcap\{\overline{f(A)}: A \in q\}$. Assume there is another element $m \in \bigcap\mathcal{B}_{q}$. Define $h: \beta\mathbb{N} \to W$ such that $h(q) = m$, which is the same way how we construct function g. Note that g and h have the same behavior from $\beta\mathbb{N} \to W$ and they will be equal if they start from $\mathbb{N}$ because the continuous functions on a dense set they are equal.

Proposition 2.2.7: Every left ideal in $\beta\mathbb{N}$ the Stone-Čech compactification for the discrete set of natural numbers $\mathbb{N}$, contains a minimal left ideal.

Proof:

By the definition of a left ideal immediately $\beta\mathbb{N} + \mathcal{P}$ is a left ideal for all $\mathcal{P} \in \beta\mathbb{N}$. Let $q \in \beta\mathbb{N} + \mathcal{P}$ implies $\beta\mathbb{N} + q \subseteq \beta\mathbb{N} + \mathcal{P}$. Note that, $\beta\mathbb{N} + q$ *and* $\beta\mathbb{N} + \mathcal{P}$ are both compact since they are images of right translation

$\rho_{\mathcal{P}}(\beta\mathbb{N})$ and $\rho_{q}(\beta\mathbb{N})$. Moreover, both are closed since $\beta\mathbb{N}$ is T_2- space and every compact subset of T_2-space is closed.

We will try to show $\beta\mathbb{N}+q$ is a minimal left ideal. Consider $\mathcal{R}=\{\beta\mathbb{N}+q_i$ a left closed ideal on $\beta\mathbb{N}$ and $\beta\mathbb{N}+q^{`} \subseteq \beta\mathbb{N}+\mathcal{P}\}$. Then $\mathcal{R}\neq\emptyset$ since we have $\beta\mathbb{N}+q_i$. So and it is partially ordered by inclusion such that $\{K_1 \subseteq K_2$ then $K_1 \leq K_2\}$. Define $\mathcal{C}=\{\beta\mathbb{N}+q_1 \supseteq \beta\mathbb{N}+q_2 \supseteq \cdots\cdots\cdots\}$ be an chain. By the finitely intersection property $\cap(\beta\mathbb{N}+q_i)\neq\emptyset$ which is a left closed ideal. Denote $S=\cap(\beta\mathbb{N}+q_i)$ which is the lower bound of $\mathcal{C}$. By Zorn's lemma $\mathcal{R}$ has a minimal left ideal $\beta\mathbb{N}+S$ among left closed ideals in $\beta\mathbb{N}+\mathcal{P}$, i.e. if $\mathcal{F}\subseteq \beta\mathbb{N}+S$ and $\mathcal{F}$ is a left closed ideal then $\mathcal{F}=S$.

To complete the proof, we need to show that S is a minimal corresponding to all space. Claim every left ideal contains a left closed ideal. Let L be a left ideal and $d\in L$. Therefor, $\rho_d(\beta\mathbb{N})=\beta\mathbb{N}+d\subseteq L$ which is an image of compact space. Furthermore, it is a closed since it's a compact subset of Hausdorff space. So, there exists $\mathcal{F}^{`}$ a left closed ideal of $L$. Now $\mathcal{F}^{`}\subseteq L\subseteq S$, i.e. $\mathcal{F}^{`}\subseteq S$ implies $\mathcal{F}^{`}=S$.

Lemma 2.2.8: The set of $M(\beta\mathbb{N})$ is an ideal in $\beta\mathbb{N}$ in fact which is the smallest ideal corresponding to $\beta\mathbb{N}$.

Proof:

$M(\beta\mathbb{N})\neq\emptyset$ since from the proposition (2.2.7) $\beta\mathbb{N}$ has a minimal left ideal. Let $\mathcal{P}\in M(\beta\mathbb{N}), \mathcal{P}\in\mathcal{F}$ which is one a minimal left ideal in $M(\beta\mathbb{N})$, then $\beta\mathbb{N}+\mathcal{P}\subseteq \mathcal{F}\subseteq M(\beta\mathbb{N})$, and so $M(\beta\mathbb{N})$ is a left ideal. Similarly, for the right ideal. Hence, $M(\beta\mathbb{N})$ is an ideal. Finally, to show $M(\beta\mathbb{N})$ is the smallest. First we need to show $M(\beta\mathbb{N})$ is a minimal ideal.

Let an ideal $L \subseteq M(\beta\mathbb{N})$ to show $L = M(\beta\mathbb{N})$. Suppose $\mathcal{F}$ be any minimal left ideal subset of $M(\beta\mathbb{N})$. Then $\mathcal{F} \cap L \neq \emptyset$.

To show $\mathcal{F} \cap L$ is a left ideal in $\beta\mathbb{N}$. Let $x \in \mathcal{F} \cap L$. Since $\beta\mathbb{N} + x \subseteq \mathcal{F}$ and $\beta\mathbb{N} + x \subseteq L$ then $\beta\mathbb{N} + x \subseteq \mathcal{F} \cap L$. Since $\mathcal{F} \cap L$ is a left ideal.

By minimalists of $\mathcal{F}$ and since $\mathcal{F} \cap L \subseteq \mathcal{F}$ then $\mathcal{F} \cap L = \mathcal{F}$

$$\Rightarrow \mathcal{F} \subseteq L$$

$$\Rightarrow L = M(\beta\mathbb{N})$$

So, $M(\beta\mathbb{N})$ is a minimal ideal.

Finally, to show $M(\beta\mathbb{N})$ is the smallest ideal . Let L be an ideal in $M(\beta\mathbb{N})$ to show $M(\beta\mathbb{N}) \subseteq L$ we know $M(\beta\mathbb{N}) \cap L \neq \emptyset$. i.e. $M(\beta\mathbb{N}) \cap L \subseteq M(\beta\mathbb{N})$. Let $x \in M(\beta\mathbb{N}) \cap L \Rightarrow \beta\mathbb{N} + x \in M(\beta\mathbb{N}) \cap L$

$\Rightarrow M(\beta\mathbb{N}) \cap L$ is an ideal

$\Rightarrow M(\beta\mathbb{N}) \cap L = M(\beta\mathbb{N})$

$\Rightarrow M(\beta\mathbb{N}) \subseteq L$.

2.3 Enfolding Semi-group

In this subsection, we define the Enfolding semi-group concept and we study topological and algebraic properties of it.

Definition 2.3.1[5]: Let K be semi-group and X be a set. Then the **action** of K on X (simply say that K acts on X) is a function:

$$\alpha : K \times X \to X$$

$(k, x) \to kx$ such that $(kt)x = k(tx)$ for all $x \in X$ and $k, t \in K$. When K has an identity say e then $e \cdot x = x$ for all $x \in X$.

Definition 2.3.2[22]: A **Dynamical System** is denoted by a pair (K, X), where X is a topological space, which is called phase space, that is defined as an acting on some group (semi-group) K on X. The set $K \cdot x = \{k \cdot x : x \in X\}$, is called the **orbit** of x. We define $\mathcal{L}_{(x)}$ to be the set of the orbit closure of $K \cdot x$.

Note that Topological Dynamics is the study of orbits for all points in X.

Now we aim to define the K-system a very important aspect of dynamical systems, which is used as a very fundamental tool in the abstract theory of Topological Dynamics.

Definition 2.3.3 [20]: A ***K*-System** is a triple (K, X, α) such that K is a semi-group and X is a Hausdorff compact space called a phase space on K- system, and $\alpha: K \times X \to X$ is a continuous action of K on X, we write $\alpha(k, x) = k \cdot x = \alpha^k(x)$.

A subset A from X in a K-system is **invariant** if $K.A = \{ta \mid a \in A, t \in K\} \subseteq A$.

Definition 2.3.4[19]: A homomorphism from two K-systems (K,X) and (K,Z) is a continuous surjection function $\alpha: X \longrightarrow Z$ satisfying $\alpha(ka) = k\alpha(a)$ for all $k \in K$ and $a \in X$. If additionally α is one to one, then it is called an **isomorphism** of K-systems**.**

Next, we will provide our main definition in this work, this will be used in the following results we give in the rest of our section.

The main concepts in our research will be the next definition.

Definition 2.3.5: Let X be a compact Hausdorff topological space and X^X denote the collection of all continuous functions from X to itself, which is a semi-group under the composition, provided with the product topology, or the topology point wise convergence. Let $T = \{f: X \to X\}$ be a subset contained in X^X . Then the closure of a set T is called an **Enfolding** semi-group of T denoted by $\mathcal{E}(T)$, given with the topology of point wise convergent. In particular, if (K, X, α) is a K-systems then the closure of the set $\{\alpha^k: k \in K\}$ in X^X is denoted by $\mathcal{E}(K, X)$ will refer to the Enfolding semi-group of the K-systems.

The motivation for studying the Enfolding semi-group is to understand the algebraic properties of a K-system.

We think the rarity and difficulty of examples of Enfolding semi-groups is that these objects are usually non-metrizable.

The following example is in [22], but it serves the desired purpose.

Example 2.3.6: For each $n \in \mathbb{N}$,

let $X_n = \{(r,\theta) = \left(\frac{1}{2^n}, \frac{2x\pi}{2^n}(mod\ 2\pi)\right):\ x = 0, 1, 2, \ldots\ldots\}$ and

$X = \bigcup_{n\in\mathbb{N}} X_n \cup \{(0,0)\}$ as a subspace of R^2.

Define $f: X \to X$ as $f(r,\theta) = (r, \theta + 2\pi r(mod\ 2\pi))$.

For any $s \in \mathbb{N}$,

$$f^s\left(\frac{1}{2^n}, \theta\right) = \left(\frac{1}{2^n}, \theta + \frac{2s\pi}{2^n}(mod 2\pi)\right).$$

Let s be a $2-$adic integer. Suppose $s = 2^{s_1} + 2^{s_2} + \cdots\cdots + 2^{s_r}$ then $f^s\left(\frac{1}{2^n}, \theta\right) = \left(\frac{1}{2^n}, \theta + 2\pi\left(\frac{2^{s_1}+2^{s_2}+\cdots+2^{s_r}}{2^n}\right)(mod\ 2\pi)\right) = \left(\frac{1}{2^n}, \theta + 2\pi\left(\frac{1}{2^{n-s_1}} + \frac{1}{2^{n-s_2}} + \cdots\cdots + \frac{1}{2^{n-s_r}}\right)(mod\ 2\pi)\right)$.

Let $a = \cdots\cdots 10101 = 1 + 4 + 16 + \ldots\ldots$ be a $2-$adic integer. Then for the function f_a defined as $f_a(r,\theta) = \left(r, \theta + 2x\pi a_n(mod(2\pi)\right)$, Where $a_{2x} = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots + \frac{1}{2^{2x-2}} + \frac{1}{2^{2x}}$ and $a_{2x+1} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2x-1}} + \frac{1}{2^{2x+1}}$. We see that f_a will be a member of $\mathcal{E}(X)$ corresponding to a.

Definition 2.3.7 [26]: A **net** in a set X is a map $S: D \to X$ where D is discrete set.

Note 2.3.8 [14]: A net S convergent to $x \in X$ if for any open set U containing x, there is $d_0 \in D$, $\forall\ d \geq d_0 : x_d \in U$.

The next Theorem it will help us to prove that the Enfolding semi-group is a right-topological semi-grou

Theorem 2.3.9: Let D be a topological space and V represent the product topology on D. Then $(D^D, \circ, V)$ is a right-topological semi-group. Furthermore, for each $h \in D^D$, h is continuous if and only if λ_h is continuous where $\lambda_h(g) = h \circ g = \rho_g(h)$.

Proof:

Suppose that $h \in D^D$, and suppose $< g_i >_{i \in I}$ is a net convergence to g in the product topology D^D. Since that, a net $< h_i >_{i \in I}$ is convergent to h in D^D if and only if $< h_i(x) >_{i \in I}$ is convergent to $h(x)$ for every $x \in D$. Implies that, $< g_i\left(h(x)\right) >_{i \in I}$ convergent to $g\left(h(x)\right)$ in D. Therefore, $< g_i \circ h >_{i \in I}$ is convergent to $g \circ h$ in D^D and therefore $g_i \circ h$ is continuous. This show ρ_h is continuous. Hence, the set of all functions from $D \to D$ is a right-topological semi-group. For the second part, suppose that h is continuous. Hence for every given net $< g_i >_{i \in I}$ convergence to g in the product topology of D^D. Then $< h(g_i(x)) >_{i \in I}$ is convergence to $h\left(g(x)\right)$ for every $x \in D$. Implies $< h \circ g_i >_{i \in I}$ is convergence to $(h \circ g)(x)$. Hence, $h \circ g_i$ is continuous, i.e. we show that λ_h is continuous.

Conversely, we suppose λ_h is continuous. Let $< x_i >_{i \in I}$ be a net converge to x in D. Define $g_i : D \longrightarrow D$ such that $g_i(x) = x_i$, $\forall\, i \in I$ and

$g : D \longrightarrow D$ such that $g(x) = x$.

Therefore, $< g_i >_{i \in I}$ converges to g in D^D so, $< h \circ g_i >_{i \in I}$ converge to $h \circ g$. This means $< h(x_i) >_{i \in I}$ converges to $h(x)$ and therefore h is continuous.

The next two results will prove that the Enfolding is a right- topological semi-group.

Proposition 2.3.10: On the right-topological semi-group D^D, let K be a subset of the topological center of D^D. If K is a semi-group then $\overline{K}$ is a semi-group.

Proof:

Let $m, n \in \overline{K}$. Let U be any neighborhood of mn. We need to show that $U \cap K \neq \emptyset$. By Theorem (2.3.9) since ρ_n is continuous, then there exist a neighborhood V of m s.t $\rho_n(V) = Vn \subseteq$ U. Since $m \in \overline{K}$, so $V \cap K \neq \emptyset$. Let $m_1 \in V \cap K$, and hence $\lambda_{m_1}(n) = m_1 n = \rho_n(m_1) \in U$. Note that $m_1 \in K \subseteq$ center D^D, which implies that $\lambda_{m_1}$is continuous. Moreover, there is a neighborhood S of n with $\lambda_{m_1}(S) \subseteq U$. Since $n \in \overline{K}$ then $S \cap K \neq \emptyset$.

Let $n_1 \in S \cap K$ and $\lambda_{m_1}(n_1) \in \lambda_{m_1}(S) \subseteq U$. Then $m_1 n_1 = \lambda_{m_1}(n_1) \in U$

$$\Longrightarrow m_1 n_1 \in K$$

$$\Longrightarrow U \cap K \neq \emptyset.$$

Hence, $mn \in \overline{K}$.

Lemma 2.3.11: The Enfolding semi-group $\mathcal{E}(T)$ is a compact right-topological semi-group.

Proof:

Since X is compact. Then by Tychonoff's theorem, $X \times X \times X$ ... is compact. But, $T \subseteq X \times X \times X$ ... and $\mathcal{E}(T)$ is a closed subset of compact Hausdorff space. Therefore, $\mathcal{E}(T) = \overline{T} \subseteq \ X \times X \ \times \ X$ ... is a compact Hausdorff space.

We need to show that ρ_f is continuous on $\mathcal{E}(T)$. Let $f \in \mathcal{E}(T)$, we need to show ρ_f is continuous on $\mathcal{E}(T)$. To do that we will first show ρ_f is continuous on T. Let $\langle g_i \rangle_{i\in I}$ be a net on T convergence to g. Since g is a pointwise convergence, then for any $x \in X, g_n(x) \longrightarrow g(x)$. Hence $g_n(f(x)) \longrightarrow g(f(x))$. Therefore ρ_f (g_n) convergence pointwise to $\rho_f(g)$. Therefore ρ_f is continuous on the dense set T and hence on its closure $\mathcal{E}(T)$.

Remark 2.3.12: The algebraic structure of the Enfolding semi-group gives some important characterization of the dynamical system properties of the K-system (K, X).

1) $\Phi : \beta K \to \mathcal{E}(K, X)$ is both a semi-group homomorphism and a K-system.

2) The map $\psi : \mathcal{E}(K, X) \to X$ defines as $t \to tx$ is a K-system homomorphism for all $x \in X$.

3) The map Φ: $\beta K \to \mathcal{E}(\beta K, K)$ is an isomorphism.

4) Let $\varphi : (K, X) \to (K, Z)$ be a homomorphism of K-system, then $\varphi(tx) = t\varphi(x)$ for all $x \in X$ and $t \in \beta K$.

Definition 2.3.13: Let (K, X, α) be a K- system and $\mathcal{E}(K, X)$ be its Enfolding semi-group. Then for a non-empty set $I \subseteq \mathcal{E}(K)$, which is **a left ideal** if $\mathcal{E}(K, X) \cdot I \subseteq I$, i.e. for $\ell \in I$ and $q \in \mathcal{E}(K, X)$ such that $q\ell \in I$. Also, I is called a **right ideal** if $\ell q \in I$. Moreover, $I \in \mathcal{E}(K, X)$ is an ideal if and only if I is both a right and a left ideal.

Definition 2.3.14: A left ideal I in an Enfolding semi-group is **minimal** if and only if I is closed in $\mathcal{E}(K,X)$ and I does not contain any other proper subset left ideal.

Lemma 2.3.15: Let $\mathcal{E}(K,X)$ be an Enfolding semi-group in K-sysytem (K,X). Then any left ideal in $\mathcal{E}(K,X)$ contains a minimal left ideal.

Proof:

Let I be any left ideal of $\mathcal{E}(K,X)$, and let $\mathcal{B} = \{J : J \text{ is a closed left ideal of } \mathcal{E}(K,X) \text{ and } J \subseteq I\}$.

Applying Zorn's lemma to $\mathcal{B}$, one gets a left ideal J minimal among all the closed left ideals that are contained in I.

Lemma 2.3.16: Let X be a compact topological space then the set of Enfolding $\mathcal{E}(K,X)$ contains an idempotent.

Proof:

Let $\mathcal{H}$ be a minimal subset of $\mathcal{E}(K,X)$ which is defined by $\{S \subseteq \mathcal{E}(K,X), S \neq \emptyset, SS \subseteq S, S \text{ is compact}\}$. Since $\mathcal{E}(K,X)$ itself satisfies these properties, so $\mathcal{H} \neq \emptyset$. We claim $\mathcal{H}$ has a minimal set of this kind. Let $\mathcal{C}$ be a chain in $\mathcal{H}$ which is a collection of closed subsets of $\mathcal{E}(K,X)$. This chain will satisfy the finite intersection-property. Therefor, $\cap\, \mathcal{C} \neq \emptyset$ is compact. Hence $\cap\, \mathcal{C} \in \mathcal{H}$. By Zorn's lemma, let A be a minimal element of $\mathcal{H}$.

We need to show $y \cdot y = y, \forall\, y \in A$, i.e. $Ay = A$. Take any $u \in A$, $Au \subseteq A$ is compact since $Au = \rho_u\,(A)$ which is a continuous image of a compact space. Let $\mathcal{B} = \{v \in A : vu = u\}$, then $\mathcal{B} \neq \emptyset$ because $u \in A = Au$ moreover, since $\mathcal{B} = A \cap \rho_u^{-1}[\{u\}]$ then $\mathcal{B}$ is closed, this implies $\mathcal{B}$ is compact. Given $x, z \in \mathcal{B}$, so $xz \in AA \subseteq A$ and $xzy = xy = y$. Implies $xz \in \mathcal{B}$, and thus, $\mathcal{B} \in \mathcal{H}$. Since $\mathcal{B} \subseteq A$ and A is minimal, thus $\mathcal{B} = A$, and hence $u \in \mathcal{B}$, and so $u \cdot u = u$.

Remark 2.3.17: Let (K, X) be K-system and $I \subseteq \mathcal{E}(K, X)$ be a minimal ideal. Then the set J of idempotent of I is non-empty.

Proposition 2.3.18: Let (K, X, α) be K-system and $I \subseteq \mathcal{E}(K, X)$ be a minimal ideal and J the set of idempotent of I then:

1) For all $q \in I$ and $v \in J$, then $vq = q$.

2) For all $v \in J$ then Iv is a group with identity v.

3) The partition of I is $\{Iv : v \in J\}$.

Proof:

1) Let $q \in I$,and $v \in J$, to prove $vq = q$. Then vI is an ideal subset of I. So $vI = I$, there exists $\mathcal{P} \in I$ with $v\mathcal{P} = q$. This implies $vq = vv\mathcal{P} = v\mathcal{P} = q$.

2) Suppose $\mathcal{P} \in Iv$. There exists $q \in I$ with $qv = \mathcal{P}$, then $\mathcal{P}v = qvv = qv = \mathcal{P}$, so v is both a left and right identity for Iv. Since I is an ideal, and $\mathcal{P}I$ is an ideal subset of I, then $\mathcal{P}I = I$. There exists $r \in I$ with $\mathcal{P}r = v$, and $\mathcal{P}(rv) = (\mathcal{P}r)v = vv = v$.

Note that $(r\mathcal{P})(r\mathcal{P}) = r(\mathcal{P}r)\mathcal{P}$

$$= r(v\mathcal{P})$$

$$= r\mathcal{P}$$

Hence, $(rv)\mathcal{P} = r\mathcal{P}$

$$= (r\mathcal{P})v = v$$

This implies rv is a left and right invers of $\mathcal{P}$ in Iv.

3) Let $q \in I$, and qI is an ideal subset of I, then $qI = I$. Suppose $S = \{\mathcal{P} \in I |\, q\mathcal{P} = q\} = \alpha_{\mathcal{P}}^{-1}(q)$ is a non-empty closed subsemi-group of I, see proof proposition (3.9) in [5].

There exists an idempotent $u \in J$ with $qu = q$, so $q \in Iu$. Then I=$\cup \{Iv |\, v \in J\}$. Let $u, v \in J$ and $q \in Iv \cap Iu$. So $q = qu = qv$ and there exists $p \in Iv$ with $pq = v$. This leads $u = vu = (pq)u = p(qu) = pq = v$.

The next proposition shows that if we have J to be the set of all idempotents in $\mathcal{E}(K, X)$. One can define an equivalence relation $\sim$ on J as $u{\sim}v$ iff $uv = v$ and $vu = u$. Then, we say u and v are equivalent.

Proposition 2.3.19: Let (K, X) be a K-system. If $I, J \subseteq \mathcal{E}(K, X)$ are minimal, ideals in $\mathcal{E}(K, X)$ and $u^2 = u \in I$ be an idempotent. Then there is $v \in J$ which is a unique idempotent with $uv = u$ and $vu = v$.

Proof:

Let $u^2 = u \in I$ and uJ is a closed ideal subset of I, this mean $uJ = I$. Suppose $A = \{\, j \in J \mid uj = u\} \neq \emptyset$. Then $A = J \cap \alpha_{u}^{-1}(u)$ is closed, and $A^2 \subseteq A$. Then there exists $v^2 = v \in A$, so $uv = u$.

Similarly, there exists $r^2 = r \in I$ with $vr = v$. We get $r = ur = uvr = uv = u$. The same way $vu = v$.

Now, suppose $\gamma^2 = \gamma \in J$ with $u\gamma = u$ and $\gamma u = \gamma$. We need to show $v = \gamma$. Then $\gamma = \gamma u = \gamma uv = \gamma v$. This means that $\gamma \in Jv \cap J\gamma$, we get $v = \gamma$, v is unique.

CHAPTER THREE

SEPARABLE POINTS IN βN BY USING MINIMAL SYSTEM

3.1 Introduction

In previous sections, we saw that the Enfolding semi-group gives several measurement of the action of $\mathbb{N}$ on X. In this chapter, we try to show that if we give two distinct points $\mathcal{P}, q \in \beta\mathbb{N}$, whether there exist some element x, in the smallest ideal in $M(\beta\mathbb{N})$ then there exists x such that $\mathcal{P} \cdot x \neq q \cdot x$. We are able to give an application of a product discrete countable space of two-point with a specific system with several conditions, some involving minimal systems, which are equivalent to ability to splitting p and q in this way. More generally, we feel that the investigation of the Enfolding semi-group structure of a minimal system is a worthy one, and interesting corresponding problems of a purely semi-group theoretic nature.

3.2 M-stenography and Minimal system

Definition 3.2.1[11]: A continuous homomorphism $\alpha: K \to H$ where K be a semi topological semi-group and H be a compact right-topological semi-group, (H, α) is called **a semi-group compactification** if $\alpha(k)$ is a dense in H and the action of K on H, $(k, t) \longrightarrow k \cdot t = \alpha(k)t : K \times H \to H$ is a continuous.

Note 3.2.2: In the case of the set of natural numbers set, $\mathbb{N}$, which is a discrete semi-group set with + operation, the semi-group compactification is the Stone-Čech compactification $\beta\mathbb{N}$ given with a semi-group additive extended from $\mathbb{N}$.

Definition 3.2.3: Let $(\mathbb{N}, X, \alpha)$ be an $\mathbb{N}$-system. Define $\varphi: \mathbb{N} \to \mathcal{E}(\mathbb{N}, X)$ be a mapping defined by $\varphi(n) = \alpha^n$, which gives us a semi-group compactification where $\alpha^n: X \to X$ for a fixed n. Thus there exist a continuous unique semi-group homomorphism $\Phi: \beta\mathbb{N} \to \mathcal{E}(\mathbb{N}, X)$ such that $\Phi \circ \psi = \varphi$.

$$\begin{tikzcd}
 & \beta\mathbb{N} \arrow[dr, "\Phi"] & \\
\mathbb{N} \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & \mathcal{E}(\mathbb{N}, X)
\end{tikzcd}$$

Remark 3.2.4:

a) Since the image of $\mathbb{N}$ is dense in $\mathcal{E}(\mathbb{N}, \mathrm{X})$, $\beta\mathbb{N}$ is compact and $\mathcal{E}(\mathbb{N}, \mathrm{X})$ is hausdorff. Thus, Φ is onto mapping.

b**)** From the definition above, the function $\Phi(n)(x) = n \cdot x$, which is a right continuous action of $\beta\mathbb{N}$ on X is called an **extended action**.

The next lemma gives an equivalence of an isomorphism between the universal semi-group compactification on the Enfolding semi-group and the extending action to be effective.

Lemma 3.2.5: The homomorphism $\Phi: \beta\mathbb{N} \to \mathcal{E}(\mathbb{N}, X)$ is a topological isomorphism if and only if the extension action is effective, i.e. if $\mathcal{P} \neq q \in \beta\mathbb{N}$, then there exists $x \in X$ such that $\mathcal{P} \cdot x \neq q \cdot x$.

Proof:

Suppose Φ is a topological isomorphism then for all $\mathcal{P} \neq q$, $\Phi(\mathcal{P}) \neq \Phi(q)$ i.e. we have $\alpha(\mathcal{P}): X \to X \neq \alpha(q): X \to X$, and this the meaning of a separating point. Thus, from extended action there exist $x \in X$ such that $\Phi(\mathcal{P})(x) \neq \Phi(q)(x)$, thus mean $\mathcal{P} \cdot x \neq q \cdot x$.

Conversely, suppose that the extension action is effective. Since the image of $\beta\mathbb{N}$ in $\mathcal{E}(\mathbb{N}, X)$ is dense. Hence, by remark above, Φ is surjective. Therefore, we have Φ is continuous surjective homomorphism.

Note that, the continuous map from compact space X to T_2-space is a closed map. Hence, Φ is a closed map sine $\{$a bijective map $f: X \to Y$ is closed iff $f^{-1}: Y \to X$is continuous$\}$.

We need only to show Φ is one to one, that is given immediately from hypothesis $\mathcal{P} \cdot x \neq q \cdot x$ for some $x \in X$ and $\mathcal{P}, q \in \beta\mathbb{N}$.

Note 3.2.6: In case $\mathcal{F}$ is a left closed ideal in the universal compactification $\beta\mathbb{N}$. Then, from the previously works we did we can define a mapping $\varphi : \mathbb{N} \to \mathcal{E}(\mathbb{N}, \mathcal{F})$ which is a semi-group compactification. Thus, there exists a continuous semi-group homomorphism $\Phi : \beta\mathbb{N} \to \mathcal{E}(\mathbb{N}, \mathcal{F})$ such that $\Phi \circ \psi = \varphi$,

$$\begin{tikzcd} & \beta\mathbb{N} \arrow[dr, "\Phi"] & \\ \mathbb{N} \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & \mathcal{E}(\mathbb{N}, \mathcal{F}) \end{tikzcd}$$

and the extended action of $\beta\mathbb{N}$ on $\mathcal{F}$ is $(k, l) = k \cdot l, k \in \mathbb{N}$, and $l \in \mathcal{F}$.

Corollary 3.2.7: Let $\beta\mathbb{N}$ be the universal compactification of the semi-group $\mathbb{N}$ and $\mathcal{F}$ be a left closed ideal of $\beta\mathbb{N}$. If $\mathcal{F}$ has a point $\mathcal{P}$ which satisfies the right cancellation. Then the homomorphism $\Phi: \beta\mathbb{N} \to \mathcal{E}(\mathbb{N}, \mathcal{F})$ is a topological isomorphism.

Proof:

By lemma (3.2.5), if we can show the extended action is effective, we are done. Let $s \neq t$ where $s, t \in \beta\mathbb{N}$. Then $s \cdot \mathcal{P} \neq t \cdot \mathcal{P}$, because otherwise if $s \cdot \mathcal{P} = t \cdot \mathcal{P}$ then by the right cancellation implies $s = t$ which is a contradiction. Hence, $\Phi(s)(\mathcal{P}) \neq \Phi(t)(\mathcal{P})$.

Lemma 3.2.8: Let K be a topological semi-group and (H, α) be a semi-group compactification of K, then there exist a left closed ideal $\mathcal{F}$ of H generating an K-system such that $(k, t) \to \alpha(k)t: K \times \mathcal{F} \to \mathcal{F}$.

Proof:

First, for the existence of the left closed ideal. Since K is a right topological semi-group then by using Corollary (1.2.24) one can have a left closed left ideal $\mathcal{F}$.

Note that, we have an action $\alpha: K \times H \to H$, which is defined by $(k, t) \to k \cdot t$ that is continuous. Moreover, $\mathcal{F}$ be a closed Hausdorff subset of a compact space H. Also $\alpha : K \times \mathcal{F} \to \mathcal{F}$ is continuous implies $\alpha|\mathcal{F} : K \times \mathcal{F} \to \mathcal{F}$ is continuous define by $\alpha(k, l) = k \cdot l$ for each $l \in \mathcal{F}$. Hence, we can define a K-system $(K, \mathcal{F}, \alpha|\mathcal{F})$.

Theorem 3.2.9: Let $\mathcal{F}$ be a left minimal ideal in a compact right-topological semi-group $\beta\mathbb{N}$ and $M(\beta\mathbb{N})$ it is the smallest ideal. Suppose $s \neq t \in \beta\mathbb{N}$. Then the following are equivalent:

a) There exists $q \in \mathcal{F}$ such that $s + q \neq t + q$.

b) There exists an idempotent $r = r + r \in F$ with $s + r \neq t + r$.

c) There exists an idempotent $e \in M(\beta\mathbb{N})$ such that $s + e \neq t + e$.

d) There is $q \in M(\beta\mathbb{N})$ such that $s + q \neq t + q$.

Proof:

From (a) to derived (b) by taking $q = r$.

From (b) → (c), immediately by taking $r = e$.

Similarly from (c) → (d) It deriving by taking $e = q$.

The implication (a) implies (d). Since $q \in \mathcal{F}$, and $\mathcal{F}$ is a minimal left ideal then $q \in M(\beta\mathbb{N})$, and by hypothesis $s + q \neq t + q$.

Now, assume (d) and derive (b): Recall the definition of the smallest ideal $M(\beta\mathbb{N}) = \cup \{\mathcal{R}$: $\mathcal{R}$ is a minimal right ideal in $M(\beta\mathbb{N})$, which implies $q \in \mathcal{R}$ for some minimal right ideal $\mathcal{R}\}$. Also by Theorem (1.28) in [19] $\mathcal{R} \cap \mathcal{F}$ is a group and therefore contains an idempotent say r.

But $\mathcal{R}$ minimal right ideal, and then $r + \mathcal{R} = \mathcal{R}$. Hence $q = r + n$ for some $n \in \mathcal{R}$.

Thus $r + q = r + r + n = r + n = q$. If $s + r = t + r$ then $s + q = s + r + q = t + r + q = t + q$ which is a contradiction. Therefore, $s + r \neq t + r$.

Definition 3.2.10: Let M be the smallest ideal of the semi-group K. The semi-group is a **right *M*-stenography** if given $s \neq t \in K$ there is $q \in M$ such that $sq \neq tq$.

Note that, the Theorem (3.2.9) gives another equivalence definition to the right M- stenography.

Definition 3.2.11[1]: The system (K, X) is called a **minimal system** if the orbit $Kx = \{k.x: k \in K\}$ is dense in X for every $x \in X$.

Definition 3.2.12: Let (H, α) be a semi-group compactification of the topological semi-group K. Then the system $(K, \mathcal{F})$ is a **minimal system** where $\mathcal{F}$ is a left closed ideal of H if and only if $\mathcal{F}$ is a minimal left ideal of H.

Theorem 3.2.13: For a topological semi-group $(\mathbb{N}, +)$ and for $(\beta\mathbb{N}, \varphi)$ be the universal semi-group compactification. The following are equivalent:

a) The semi-group $\beta\mathbb{N}$ is right M- stenography.

b) If $\mathcal{F}$ is a minimal left ideal of $\beta\mathbb{N}$, then the homomorphism for the minimal system $(\mathbb{N}, \mathcal{F})$ is an isomorphism, and therefore the Enfolding semi-group of this minimal system is topologically isomorphic to $\beta\mathbb{N}$.

c) Given $\mathcal{P}, q \in \beta\mathbb{N}$ such that $\mathcal{P} \neq q$, there exist a minimal system $(\mathbb{N}, X)$ and $x \in X$ such that $\mathcal{P} \cdot x \neq q \cdot x$ regarding to the extended action.

Proof:

$a) \Longrightarrow b)$: Let $\mathcal{F}$ be a left minimal ideal of $\beta\mathbb{N}$. To show $\Phi: \beta\mathbb{N} \longrightarrow \mathcal{E}(\mathbb{N}, \mathcal{F})$ is isomorphism. If we can show Φ is effective then by lemma (3.2.5) we are done. Let $\mathcal{P} \neq q \in \beta\mathbb{N}$ and since $\beta\mathbb{N}$ is right M-stenography then $\exists\, x \in \mathcal{F}$ s.t $\mathcal{P} \cdot x \neq q \cdot x$.

$b) \Longrightarrow c)$: We have $\mathcal{F}$ is a minimal left ideal in a compact right topology $\beta\mathbb{N}$. So by corollary (1.2.24) $\mathcal{F}$ is closed. Then $(\mathbb{N}, \mathcal{F})$ is a minimal system by definition (3.2.12). But, we have $\Phi: \beta\mathbb{N} \longrightarrow \mathcal{E}(\mathbb{N},\mathcal{F})$ is isomorphism and so by lemma (3.2.5) the action is effective.

$c) \Longrightarrow a)$: Let $\mathcal{P} \neq q \in \beta\mathbb{N}$ and $(\mathbb{N}, X)$ be a minimal system such that for some $\in X$, $\mathcal{P} \cdot x \neq q \cdot x$. Let $\mathcal{F}$ be a minimal left ideal in $\beta\mathbb{N}$ implies $\mathcal{F}$ is closed by corollary (1.2.24). Then $\mathcal{F} \cdot x \in X$ is a closed since it is a continuous image of closed left ideal. Moreover, since $\mathbb{N}(\mathcal{F} \cdot x) = (\mathbb{N} \cdot \mathcal{F}) \cdot x = \mathcal{F} \cdot x$ then $\mathcal{F} \cdot x$ is an invariant under $\mathbb{N}$. Hence, $\mathcal{F} \cdot x$ is Dense since $(\mathbb{N}, X)$ is a minimal system. Thus, $t \cdot x = x$ for some $t \in \mathcal{F}$.

Therefore, $\mathcal{P} \cdot t = q \cdot t$ implies $\mathcal{P} \cdot x = \mathcal{P} \cdot t \cdot x = q \cdot t \cdot x = q \cdot x$ which is a contradiction with our hypothesis. Hence $\mathcal{P} \cdot t \neq q \cdot t$ and thus required also $\beta\mathbb{N}$ is a right M- stenography.

3.3 Splitting points of $\boldsymbol{\beta}\mathbb{N}$ in $\boldsymbol{M(\beta}\mathbb{N})$

Let $\mathbb{N}$ be a topological semi-group with universal semi-group compactification $\beta\mathbb{N}$. According to Theorems (3.2.13) and (3.2.9), we want to know that: given $\mathcal{P} \neq \mathcal{t} \in \beta\mathbb{N}$, whether there is some $q \in M(\beta\mathbb{N})$, the smallest ideal of $\beta\mathbb{N}$, such that $\mathcal{P}q \neq \mathcal{t}q$.

The next two following definitions are in [7] and [9], but we will define them in the set of natural number $\mathbb{N}$.

Definition 3.3.1[9]: Let (K, X) be a system. A point $x \in X$ is a **uniformly recurrent point** if given any neighbourhood V of x, there is a finite compact subset M of K such that given $k \in K$, there is $m \in M$ with $mkx \in V$.

Definition 3.3.2: Let (K, X) be a system. A point $x \in X$ is an **almost recurrent point** if given any neighbourhood V of x, there is a compact subset M of K such that given $k \in K$, there exist $m \in M$ with $mkx \in V$.

Remark 3.3.3: The uniformly recurrent points it will be exactly the points that are almost recurrent for the system $(\mathbb{N}, X)$ when $\mathbb{N}$ is given with the discrete topology.

The next theorem overlapping with studying the separable points of $\beta\mathbb{N}$ in $\mathbb{N}$.

Theorem 3.3.4: Let $(\mathbb{N}, X)$ be a system, $\mathcal{P} \in X$, and $\mathcal{F}$ are minimal left ideals in the universal semi-group compactification $\beta\mathbb{N}$. The following are equivalent:

a) The orbit closure $\mathcal{L}_{(\mathcal{P})}$ contains $\mathcal{P}$ and is a minimal $\mathbb{N}$-system.

b) For the extended action of $\beta\mathbb{N}$ on X, there is $m \in M$,such that the smallest ideal in $\beta\mathbb{N}$, and $x \in X$ such that $\mathcal{P} = m \cdot x$.

c) $\mathcal{P} \in \mathcal{F} \cdot \mathcal{P}$.

d) There is e an idempotent $e \in \mathcal{F}$ such that $e \cdot \mathcal{P} = \mathcal{P}$.

e) $\mathcal{L}_{(\mathcal{P})} = \mathcal{F} \cdot \mathcal{P}$.

f) The point $\mathcal{P}$ is an almost recurrent point.

g) The point $\mathcal{P}$ is an uniformly recurrent.

Proof:

The equivalence of the first five parts can be used Theorem (2.38) in [19] by substitution $\mathbb{N}$ and $\beta\mathbb{N}$. In addition equivalence (f) can be looked to Theorem (4.2) in [16]. The equivalence of (g) can get by taking $\mathbb{N}$ with the discrete space.

In the following Theorem we consider $\beta\mathbb{N}^* = \beta\mathbb{N} \cup \{0\}$ where $\{0\}$ is the identity of the natural number $(\mathbb{N}, +)$.

Theorem 3.3.5: Let $\beta\mathbb{N}$ be the universal compactification of the topological semi-group $\mathbb{N}$. Let $\mathcal{P} \neq \mathcal{t} \in \beta\mathbb{N}$ and $\mathcal{F}$ be a minimal left ideal of $\beta\mathbb{N}$. The following statements are equivalent:

a) There exist $q \in \mathcal{F}$ such that $\mathcal{P}q \neq \mathcal{t}q$.

b) There is a system $(\mathbb{N}, X)$ and an almost recurrent (or an uniformly recurrent) point $x \in X$ such that $\mathcal{P} \cdot x \neq \mathcal{t} \cdot x$ for the extended action.

c) Let $(\mathbb{N}, W)$ be any system for which the homomorphism from $\beta\mathbb{N} \longrightarrow \mathcal{E}(\mathbb{N}, W)$ is an isomorphism. Then there is an almost recurrent (or uniformly recurrent) point $w \in W$ such that $\mathcal{P} \cdot w \neq \mathcal{t} \cdot w$.

Proof:

a) => b): Since $\mathcal{F}$ be a minimal left ideal of $\beta\mathbb{N}$ then we can consider $(\mathbb{N}, \mathcal{F})$ be a minimal system. By hypothesis $\mathcal{P}q \neq \mathcal{t}q$ and $\mathcal{F}$ is minimal this implies

$\mathcal{F}.q = \mathcal{F}$, so $q \in \mathcal{F}.q$. Then by Theorem (3.3.4) part (f) and (g), implies q is an uniformly recurrent and an almost recurrent point.

b) => a): Let $(\mathbb{N}, X)$ be a system, and let x be an almost recurrent (or an uniformly recurrent) point in X such that $\mathcal{P} \cdot x \neq \mathcal{t} \cdot x$. By Theorem (3.3.4), $x \in \mathcal{F} \cdot x$. Hence, there exist $q \in \mathcal{F}$ such that $x = q \cdot x$.

Then $\mathcal{P}q \cdot x = \mathcal{P} \cdot q \cdot x$

$$= \mathcal{P} \cdot x$$

$$\neq \mathcal{t} \cdot x$$

$$= \mathcal{t} \cdot q \cdot x$$

$$= \mathcal{t}q \cdot x$$

Hence, $\mathcal{P}q \neq \mathcal{t}q$.

a) => c): Let $(\mathbb{N}, W)$ be a system for which the homomorphism is an isomorphism. By Lemma (3.2.5), there exist $w \in W$ such that $\mathcal{P} \cdot q \cdot w = \mathcal{P}q \cdot w \neq \mathcal{t}q \cdot w = \mathcal{t} \cdot q \cdot w$, and by Theorem (3.3.4) (since condition (b) holds implies (f) and (g) is hold) then point $q \cdot w$ is an almost recurrent and an uniformly recurrent.

c) => a): Consider the system $(\mathbb{N}, \beta\mathbb{N}^*)$ of the universal semi-group compactification $(\beta\mathbb{N}, \psi)$, where $\{0\}$ is a discrete point added to $\beta\mathbb{N}$, and $n + 0 = \psi(n)$ for all $n \in \mathbb{N}$. The addition of $\beta\mathbb{N}$ extends to $\beta\mathbb{N}^*$ by making $\{0\}$ act as an identity of $\beta\mathbb{N}$, and $\beta\mathbb{N}^*$ keep it a right topological semi-group.

Then for $\mathcal{P} \neq t \in \beta\mathbb{N}$, we have $\mathcal{P} + o = \mathcal{P} \neq t = t + o$. Hence, by Lemma (3.2.5), the homomorphism $\Phi: \beta\mathbb{N} \longrightarrow \mathcal{E}(\mathbb{N}, \beta\mathbb{N}^*)$ is an isomorphism.

From hypothesis, there is an almost recurrent point q in $\beta\mathbb{N}^*$ such that $\mathcal{P}q \neq tq$. Since $\mathcal{F}$ is a minimal immediately by Theorem (3.3.4), $q \in \mathcal{F}q$. Thus, there is $v \in \mathcal{F}$ such that $q = vq$. Then $\mathcal{P}vq = \mathcal{P}q \neq tq = tvq$, and so $\mathcal{P}v \neq tv$.

3.4 Application example of separating of $\beta\mathbb{N}$

In this section and for our objective we give an application for a specific example of a semi-group and its compactification with a particular system. We will introduce a specific subset example $Y = \{0,1\}^{\mathbb{N}}$ defined with a shifting operator with a specific system. We will apply some facts we discussed in the previous section on a set Y and conclude some properties.

Example 3.4.1: Let $Y = \{0,1\}^{\mathbb{N}}$ be the countable of product discrete space consisting from two-points. We can think about a set Y as: $Y = \{\chi : N \to \{0,1\}\}$ where χ represented the characteristic function and each members x of Y can be viewed as infinite tuples like $x = (\chi(1), \chi(2), \chi(3), \dots)$.

Now define the shift operator $T: Y \longrightarrow Y$ by $T(x)(n) = \chi(n + 1)$where $x \in Y$ is shifting a tuple to the left one place, ignoring the first entry.

We consider the set of semi-groups of continuous functions $\{T^n: n \in \mathbb{N}\}$. Let $\omega = \mathcal{E}(\{T^n : n \in \mathbb{N}\}, Y)$ be the Enfolding semi-group set.

Let $\varphi: \mathbb{N} \longrightarrow \mathcal{E}(\{T^n : n \in \mathbb{N}\}, Y)$ by $\varphi(n) = T^n$, and the extension of φ denote by $\Phi: \beta\mathbb{N} \longrightarrow \mathcal{E}(\{T^n: n \in N\}, Y)$ be a homomorphism.

The next theorem result is a modification of the result of reference [7].

Theorem 3.4.2: The homomorphism $\Phi: \beta\mathbb{N} \longrightarrow \omega =: \mathcal{E}(\{T^n : n \in \mathbb{N}\}, Y)$ is defined in the example above is a topological isomorphism.

Proof:

The essence of a proof of the theorem will be by using lemma (3.2.5). Let $q \neq \mathcal{P}$ be two ultrafilters in $\beta\mathbb{N}$. Note that the points in $\mathbb{N}$ are separated by Φ since each n gives a different T^n.

Therefore, Φ is separates points on $\mathbb{N}$ and this show the separation of the case of a principle ultra-filter. For generality suppose without losing the generality assume that q is a non-principal ultra-filter. Then there exists an infinite set $B \subseteq \mathbb{N}$ such that $B \in q$ and since q and $\mathcal{P}$ are distinct then its complement $B^c \in \mathcal{P}$. Define the set $\mathcal{C} = B + 1$, and let $x \in Y$ and $\chi_{\mathcal{C}}$ be the characteristic function of $\mathcal{C}$.

For each $n \in B, \Phi(n)(x) = T^n(x)$ and $T^n(x)(1) = \chi(n+1) = 1$ since $n + 1 \in \mathcal{C}$. But $B \in q$ implies $q \in \overline{B} = \hat{B}$.

Hence $q \cdot x(1) = \big(\Phi(q)(x)\big)(1)$

$$= T^n(x)(1)$$

$$= \chi(n+1)$$

$$= 1$$

Similarly if $n \notin B$, then $\mathcal{P} \cdot x(1) = (\Phi(\mathcal{P})(x))(1)$

$$= T^n(x)(1)$$

$$= \chi(n+1) = 0$$

Thus $q \cdot x \neq \mathcal{P} \cdot x$, and hence by Lemma (3.2.5) implies Φ is topological isomorphism.

Remark 3.4.3:

a) From above work a separable by appoint $q \in \beta\mathbb{N}$ means that for any two distinct element $\mathcal{P}, t \in \beta\mathbb{N}$ then $\mathcal{P} \cdot q \neq t \cdot q$, and this q satisfy that for any other two distinct element.
b) By using Theorem (3.4.2) and **(**3.3.5) one can get a separable by a minimal left ideal $\mathcal{F}$ but this required a one can find a uniformly recurrent point in $\beta\mathbb{N}$.

Remark 3.4.4: The uniformly recurrent points can also be represented as the characteristic functions χ_A on $\mathbb{N}$ for some $A \subseteq \mathbb{N}$, that are almost periodic functions. This because we can pick any basic neighbourhood and on that neighbourhood, we assume there is a compact set K in that neighbourhood such that when we do translation with this characteristic functions on these points on K will get this neighbourhood.

For example if $A = \{4,8,16, \dots\} \subseteq \mathbb{N}$ then $y = \chi_A = (0,0,0,1,0,0,0,1, \dots)$ and let $U = \{0\} \times \{0\} \times \{0\} \times \{1\} \times Y \times Y \dots$ be a basic neighborhood of y. Note that, with the almost periodic function then

$$T(y) \notin U$$

$$T^2(y) \notin U$$

$$T^3(y) \notin U$$

$$T^4(y) \in U$$

Lemma 3.4.5: Suppose $q \in \beta\mathbb{N}$ and $x \in Y$, $n \in \mathbb{N}$. Then $\Phi(q)(x)(n) = 1$ if and only if $x^{-1}(\{1\}) - n \in q$ such that $x = \chi_B$ where $B = \{n : \chi(n) = 1\}$.

Proof:

Let، $q \in \beta\mathbb{N}$, $x \in Y$and $n \in \mathbb{N}$ i.e $x = \{\chi : \mathbb{N} \longrightarrow \{0,1\}\}$ and so $x = (\chi(n_1), \chi(n_2), \ldots)$. Suppose that $B = x^{-1}[\{1\}] \subseteq \mathbb{N}$ and $j = \Phi(q)(x)(n)$ where $\Phi : \beta\mathbb{N} \longrightarrow \omega =: \mathcal{E}(\{T^n : n \in \mathbb{N}\}, Y)$, note $\Phi(q)(x) \in Y = \{\chi : \mathbb{N} \longrightarrow \{0,1\}\}$, and $\Phi(q)(x)(n) = 0$ or 1, and let $V = \{y \in Y : y(n) = j\}$ be a neighborhood of $\Phi(q)(x)$. So that $U = \{f \in \mathcal{E} : f(x) \in V\}$ is a neighborhood of $\Phi(q)$.

$$\begin{tikzcd} & \beta\mathbb{N} \arrow[dr, "\Phi"] & \\ \mathbb{N} \arrow[ur, "\psi"] \arrow[rr, "\varphi"'] & & \omega \end{tikzcd}$$

Note that, from the diagram $\Phi(q) \in \omega$.

Therefore, $\Phi(q)(x) \in V$ from definition of V. Note that, the element in ω is set of continues functions $f : Y \longrightarrow Y$ and the topologic define on ω will be the product topology so we have a function $g : \omega \longrightarrow Y$ is a continuous function since its just a projection map. Such that, the evaluation at x we have $g(f) = f(x) \in Y$ since V is a neighborhood of $\Phi(q)(x)$ and $g^{-1}(V) = U$ is a neighborhood of $\Phi(q)$ since g is continuous. Note that, $\Phi(q) \in U$ and $\Phi^{-1}(U)$ is an open set containing q. Hence, there exist a basic open set $\hat{A} = \bar{A}$ such that $q \in \hat{A} \subseteq \Phi^{-1}(U)$. This leads to $q \in \bar{A} = \hat{A}$ is a basic neighborhood and since Φ is a closed map then $\Phi(\bar{A}) = \overline{\Phi(A)}$ and since U is a neighborhood of $\Phi(q)$ so we have an open set $\Phi(\bar{A})$ subset of U. Now pick $A \in q$ such that $\Phi[\bar{A}] \subseteq U$.

We claim if $j = 0$ then $A \cap (B - n) = \emptyset$ this implies $A \cap (B - n) \notin q$. Now, if $j = 1$ then $A \subseteq B - n$ and since $A \in q$ this implies $B - n \in q$.

Let $m \in A$ be given then $T^m = \Phi(m) \in U$ since $\Phi[\bar{A}] \subseteq U$. So that from definition of U then $T^m(x) \in V$ and $T^m(x)(n) = j$. That is $\chi(m + n) = j$. So if $j = 0$ then $m + n \notin B$. If $j = 1$ then $m + n \in B$.

The next two following statements $\Gamma(B)$ and $\Omega(B)$ in the next remark both work to characterize our problem of uniformly recurrent on $\{0.1\}^{\mathbb{N}}$.

Remark 3.4.6: Let $B \subseteq \mathbb{N}$; we will use the following statements representing:

a) $\Gamma(B)$ is the phrase that χ_B is the uniformly recurrent on $\{0.1\}^{\mathbb{N}}$.

b) $\Omega(B)$ is the phrase that there is a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{N}$ and a sequence $\{m(n)\}_{n=1}^{\infty}$ in $\mathbb{N}$ such that

i. $(\cup_{n \in B} A_n + n) \cap \left(\cup_{n \in \mathbb{N} \setminus B} A_n + n \right) = \emptyset$

ii. For every $n \in \mathbb{N}$, $\mathbb{N} = \cup_{t=1}^{m(n)} (A_n - t)$, and

iii. For every $n \in \mathbb{N}$, $A_{n+1} \subseteq A_n$.

Lemma 3.4.7: Let $B \subseteq \mathbb{N}$ and if statement $\Omega(B)$ is hold then $\Gamma(B)$ is hold.

Proof:

From definition of $\Omega(B)$, one can pick $< A_n >_{n=1}^{\infty}$ and $< m(n) >_{n=1}^{\infty}$. Let $y = \chi_B$ and let V be a neighborhood of y in Y. We assume that we have some $b \in \mathbb{N}$ such that $V = \{x \in Y$:for each $j \in \{1,2,3, \ldots, b\},\ x(j) = y(j)\}$.

i.e. for instance if $y = \chi_B = (0,1,1,0,1,0,\ldots)$ and $V = \{0\} \times \{1\} \times \{1\} \times Y \times Y \times \cdots$ be a neighbourhood of y and let $b = 3$

$y_1 = (0,1,1,1,1, \dots.)$
$y_2 = (0,1,1,0,0, \dots.) \quad \in V$
$y_3 = (0,1,1,1,0, \dots.)$

We will show that it for every $r \in \mathbb{N}$ there exists $d \in \{1, 2, 3, \dots, m(b)\}$ with $T^{r+d}(y) \in V$. Let $r \in \mathbb{N}$ be given and let $k = r + m(b) + b$. Pick some $h \in A_k$. Now from the Remark above part (2), $h + r \in \cup_{d=1}^{m(b)} A_b - d$ implies $h + r \in A_b - d$ for some d. So pick $d \in \{1,2,3,\dots\dots, m(b)\}$ with $h + r + d \in A_b$. Suppose $T^{r+d}(y) \notin V$ i.e. is not a recurrent point corresponding to d. So pick $j \in \{1,2,3,\dots.,b\}$ such that $T^{r+d}(y)(j) \neq y(j)$. Let $a = r + d + j$. Since by definition $T^{r+d}(y)(j) = y(r + d + j)$ then $y(j) \neq y(a)$ so either $a \in B$ and $j \notin B$ or $a \notin B$ and $j \in B$.

In both cases from the Remark above part (2) we have $(A_a + a) \cap (A_j + j) = \emptyset$. But $h \in A_k \subseteq A_a$ (since $a \leq k$ and $d \leq m(b)$ and $a = r + d + j \leq k = r + m(b) + b$). Therefore $h + a \in A_a + a$. Also $h + a - j = h + r + d \in A_b \subseteq A_j$. Hence, $h + a \in A_j + j$ is a contradiction since $(A_a + a) \cap (A_j + j) = \emptyset$.

Lemma 3.4.8: Let $B \subseteq \mathbb{N}$ and suppose $\Gamma(B)$ then $\Gamma(B + 1)$ or $\Gamma((B + 1) \cup \{1\})$ is hold.

Proof:

Let $x = \chi_B$ be a uniformly recurrent, by the definition of $\Gamma(B)$ for each $t \in \mathbb{N}$, let $v_t = \{y \in Y$:for all $j \in \{1,2,3, \dots, t\}, y(j) = x(j)\}$.

So we can have $\{\exists n \in \mathbb{N}: T^n(x) \in v_t\} \neq \emptyset$. Thus we can pick $\ell(t) \in \mathbb{N}$ such that $T^{\ell(t)}(x) \in v_t$.

Case (1): If $\{t \in \mathbb{N}: \ell(t) \in B\}$ is infinite. Let $A = (B+1) \cup \{1\}$ to show $\Gamma(A)$ is hold. Let $y = \chi_A$. And let $u_t = \{z \in Y: \text{for all } j \in \{1,2,3,\dots,t\}, z(j) = y(j)\}$ be the one of these v_t. Which we constructed from above which is a neighbourhood of χ_B. Since we have infinite set, pick $t^` > t$ such that $\ell(t^`) \in B$. Pick $r \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ $\exists$ some $d \in \{1,2,3,\dots,r\}$ satisfies $T^{n+d}(x) \in v_{\ell(t^`)+t}$. To show that for all $n \in \mathbb{N}$, some $d \in \{1,2,3,\dots,r+\ell(t^`)\}$ has $T^{n+d}(y) \in u_t$.

Let $n \in \mathbb{N}$ and take $j \in \{1,2,3,\dots,r\}$ such that $T^{n+j}(x) \in v_{\ell(t^`)+t}$ from hypothesis. Let $d = j + \ell(t^`)$. Then $d \in \{1,2,3,\dots,r+\ell(t)\}$.

We claim that $T^{n+d}(y) \in u_t$. To this end, let $i \in \{1,2,3,\dots,t\}$ be given. Assume $i = 1$ then $y(i) = \chi_A(1) = 1$ since $1 \in A$, and $T^{n+d}(y)(i) = y(n+d+1)$

$$\begin{aligned} &= \chi_A(n+d+1) \\ &= \chi_B(n+d) \\ &= x(n+d) \\ &= x\left(n+j+\ell(t^`)\right) \end{aligned}$$

Now $T^{(n+j)}(x) \in v_{\ell(t^`)+t}$ , and so $T^{(n+j)}(x)\left(\ell(t^`)\right) = x\left(\ell(t^`)\right) = 1$ since $\ell(t^`) \in B$. That is $T^{n+d}(y)(i) = y(i)$.

Now, assume $i \in \{2,3,\dots,t\}$, then $y(i) = x(i-1) = \chi_A(i-1)$, and $T^{n+d}(y)(i) = y(n+d+i)$

$$\begin{aligned} &= x(n+d+i-1) \\ &= x(n+j+\ell(t^`)+i-1) \end{aligned}$$

Again $T^{n+j}(x) \in v_{\ell(t^{`})+t}$ so, $T^{n+j}(x)\big(\ell(t^{`}) + i - 1\big) = x\big(\ell(t^{`}) + i - 1\big)$. Since $T^{\ell(t^{`})}(x) \in v_{t^{`}}$, then $x\big(\ell(t^{`}) + i - 1\big) = T^{\ell(t^{`})}(x)(i-1) = x(i-1)$. Hence, $T^{n+d}(y)(i) = y(i)$.

Case (2): If $\{t \in \mathbb{N}: \ell(t) \in B\}$ is finite. Then $\{t \in \mathbb{N}: \ell(t) \notin B\}$ is infinite. Assume $A = B + 1$ and the proof is similar as case (1).

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الخلاصة

الهدف الرئيسي المراد في هذا العمل تم من خلال عرض التعقيدات الكاملة لمفهوم منطوي شبه الزمرة يمكن أن تظهر في نظام أصغري. ونرى أن الدراسة في هيكل المنطوي شبه الزمرة في لنظام الأصغري هو موضوع جدير بالاهتمام في دراسة بعض المشكلات ذات أهمية للباحثين في طبيعة النظرية البحتة لشبه الزمرة. تمت دراسة المثاليات اليسارية الاصغرية $\mathcal{F}$ في فضاء التراص شبه الزمرة الشامل βK لشبه الزمرة التبولوجية K. تم البرهنة على أن المنطوي لشبه الزمرة $\mathcal{F}$ متماثل الشكل إلى βK إذا وفقط إذا أعطيت $q \neq r$ في βK ، فيمكن ايجاد عنصر p في المثالي الاصغري في الفضاء βK يحقق ان $q \cdot p \neq r \cdot p$. نشتق العديد من الشروط ، بعضها تضمن حول النظام الاصغري والذي يمكننا القدرة على فصل النقاط q و r باستخدام هذه الشروط. ومن خلال أخذ حالة خاصة والتي يكون فيها شبة الزمرة $K = \mathbb{N}$، وفضاء التراص هو $\beta\ \mathbb{N}$ تم اعطاء تطبيقًا بأخذ مجموعة قابلة للعد لنظام تكراري من الدوال مكونة من عديدين مع نظام محدد بشروط متعددة وتم دراسة النظام الادنى وكيفية فصل النقاط q و r بهذه الطريقة التي تمت دراستها.

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